

BOUNDARY VALUE PROBLEMS ON PLANAR GRAPHS AND FLAT SURFACES WITH INTEGER CONE SINGULARITIES, I: THE DIRICHLET PROBLEM

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ABSTRACT. Consider a planar, bounded, m -connected region Ω , and let $\partial\Omega$ be its boundary. Let \mathcal{T} be a cellular decomposition of $\Omega \cup \partial\Omega$, where each 2-cell is either a triangle or a quadrilateral. From these data and a conductance function we construct a canonical pair (S, f) where S is a genus $(m - 1)$ *singular flat surface* tiled by rectangles and f is an energy preserving mapping from $\mathcal{T}^{(1)}$ onto S . By a singular flat surface, we will mean a surface which carries a metric structure locally modeled on the Euclidean plane, except at a finite number of points. These points have cone singularities, and the cone angle is allowed to take any positive value (see for instance [28] for an excellent survey). Our realization may be considered as a discrete uniformization of planar bounded regions.

0. INTRODUCTION

Consider a planar, bounded, m -connected region Ω , and let $\partial\Omega$ be its boundary. Let \mathcal{T} be a cellular decomposition of $\Omega \cup \partial\Omega$, where each cell is either a triangle or a quadrilateral. Let $\partial\Omega = E_1 \sqcup E_2$, where E_1 is the outermost component of $\partial\Omega$. Invoke a *conductance function* on $\mathcal{T}^{(1)}$, making it a *finite network*, and use it to define a combinatorial Laplacian Δ on $\mathcal{T}^{(0)}$. Let k be a positive constant. Let g be the solution of a *Dirichlet boundary value problem* (D-BVP) defined on $\mathcal{T}^{(0)}$ and determined by requiring that $g|_{E_1} = k, g|_{E_2} = 0$ and $\Delta g = 0$ at every interior vertex of $\mathcal{T}^{(0)}$, i.e. g is *combinatorially harmonic*. Furthermore, let $E(g)$ be the *Dirichlet energy* of g . Following the notation in [13], let $\frac{\partial g}{\partial n}(x)$ denote the *normal derivative* of g at the vertex $x \in \partial\Omega$. Our first result is:

Theorem 0.1. (Discrete uniformization of a pair of pants) *Let \mathcal{P} be a bounded, triply connected planar region and let $(\Omega, \partial\Omega, \mathcal{T}) = (\mathcal{P}, \partial\mathcal{P} = E_1 \sqcup E_2, \mathcal{T})$ where $E_2 = E_2^1 \sqcup E_2^2$. Let $S_{\mathcal{P}}$ be a singular flat pair of pants with exactly one singular point Q having 4π as its cone angle such that*

- (1) $\text{Length}_{\text{Euclidean}}(S_{\mathcal{P}})_{E_1} = \sum_{x \in E_1} \frac{\partial g}{\partial n}(x),$
- (2) $\text{Length}_{\text{Euclidean}}(S_{\mathcal{P}})_{E_2^1} = - \sum_{x \in E_2^1} \frac{\partial g}{\partial n}(x),$
- (3) $\text{Length}_{\text{Euclidean}}(S_{\mathcal{P}})_{E_2^2} = - \sum_{x \in E_2^2} \frac{\partial g}{\partial n}(x),$ and

(4) *the Euclidean length of a shortest geodesic connecting E_1 to either E_2^1 or E_2^2 is k , where $(S_{\mathcal{P}})_{E_1}, (S_{\mathcal{P}})_{E_2^1}$ and $(S_{\mathcal{P}})_{E_2^2}$ are the boundary components of $S_{\mathcal{P}}$. Then there exists a mapping f which associates to each edge in $\mathcal{T}^{(1)}$ a unique Euclidean rectangle in $S_{\mathcal{P}}$, in such a way that the collection of these rectangles forms a tiling of $S_{\mathcal{P}}$. Furthermore, f is energy preserving in the sense that $E(g) = \text{Area}(S_{\mathcal{P}})$, and f is boundary preserving.*

Throughout this paper a Euclidean rectangle will denote the image under an isometry of a planar Euclidean rectangle. For instance, most of the image rectangles that we will construct embed in a flat Euclidean cylinder. These will further be glued in a way that will not distort the Euclidean structure (see §3 and §4 for the details). In our setting, boundary preserving means that the rectangle associated to an edge $[u, v]$ with $u \in \partial\Omega$ has one of its edges on a corresponding boundary component of the singular surface.

A singular flat, genus zero compact surface with $m \geq 3$ boundary components with conical singularities, will be called a *ladder of singular pairs of pants*. We now state the main result of this paper:

Theorem 0.2. (The general case) *Let $(\Omega, \partial\Omega, \mathcal{T})$ be a bounded, m -connected, planar region with $E_2 = E_2^1 \sqcup E_2^2 \dots \sqcup E_2^{m-1}$. Let S_Ω be a ladder of singular pairs of pants such that*

- (1) $\text{Length}_{\text{Euclidean}}(S_\Omega)_{E_1} = \sum_{x \in E_1} \frac{\partial g}{\partial n}(x)$, and
- (2) $\text{Length}_{\text{Euclidean}}(S_\Omega)_{E_2^i} = - \sum_{x \in E_2^i} \frac{\partial g}{\partial n}(x)$, for $i = 1, \dots, m-1$,

where $(S_\Omega)_{E_1}$ and $(S_\Omega)_{E_2^i}$, for $i = 1, \dots, m-1$, are the boundary components of S_Ω . Then, there exists a mapping f which associates to each edge in $\mathcal{T}^{(1)}$ a unique Euclidean rectangle in S_Ω in such a way that the collection of these rectangles forms a tiling of S_Ω . Furthermore, f is boundary preserving, and f is energy preserving in the sense that $E(g) = \text{Area}(S_\Omega)$.

The following Corollary is straightforward, thus establishing the statement in the abstract of this paper:

Corollary 0.3. *Under the assumptions of Theorem 0.2, there exists a canonical pair (S, f) , where S is a flat surface with conical singularities of genus $(m-1)$ tiled by Euclidean rectangles and f is an energy preserving mapping from $\mathcal{T}^{(1)}$ into S , in the sense that $2E(g) = \text{Area}(S)$. Moreover, S admits a pair of pants decomposition whose dividing curves have Euclidean lengths given by (1) – (3) of Theorem 0.1.*

Proof. Given $(\Omega, \partial\Omega, \mathcal{T})$, glue together two copies of S_Ω (their existence is guaranteed by Theorem 0.2) along corresponding boundary components. This results in a flat surface $S = S_\Omega \bigcup_{\partial\Omega} S_\Omega$ of genus $(m-1)$ and a mapping \bar{f} which restricts to f on each copy. □

In the course of the proofs of Theorem 0.1 and Theorem 0.2, it will become apparent that the number of singular points and their cone angles, as well as the lengths of shortest geodesics between boundary curves in the ladder, may be explicitly determined. In particular, the cone angles obtained by our construction are always even integer multiples of π (see Equation (4.12) for the actual computation). Some of these surfaces, those with trivial monodromy, are called *translation surfaces* and for excellent accounts see for instance [21], [25] and [30]. Also, the dimensions of each rectangle are determined by the given D-BVP problem on $\mathcal{T}^{(0)}$. For $[u, v] \in \mathcal{T}^{(1)}$ the associated rectangle will have its height equal to $(g(u) - g(v))$ and its width equal to $c(u, v)(g(u) - g(v))$, when $g(u) > g(v)$. Some of the rectangles constructed are not embedded. We will comment on this point (which is also transparent in the proofs) in Remark (4.15). In a snapshot, some of the rectangles which arise from intersection of edges with singular level curves are not embedded.

The following theorem is foundational and serves as a building block in the proofs of all of the above theorems. We prove:

Theorem 0.4. (Discrete uniformization of an annulus [10]) *Let \mathcal{A} be an annulus and let $(\Omega, \partial\Omega, \mathcal{T}) = (\mathcal{A}, \partial\mathcal{A} = E_1 \sqcup E_2, \mathcal{T})$. Let $S_{\mathcal{A}}$ be a straight Euclidean cylinder with height $H = k$ and circumference*

$$C = \sum_{x \in E_1} \frac{\partial g}{\partial n}(x).$$

Then there exists a mapping f which associates to each edge in $\mathcal{T}^{(1)}$ a unique embedded Euclidean rectangle in $S_{\mathcal{A}}$ in such a way that the collection of these rectangles forms a tiling of $S_{\mathcal{A}}$. Furthermore, f is boundary preserving, and f is energy preserving in the sense that $E(g) = \text{Area}(S_{\mathcal{A}})$.

Given $(\Omega, \partial\Omega, \mathcal{T})$, we will work with the natural affine structure induced by the cellular decomposition. Let us denote this cell complex endowed with this affine structure by $\mathcal{C}(\Omega, \partial\Omega, \mathcal{T})$. There is a common thread in our proofs of the above theorems. Extend g to an affine map \bar{g} defined on \mathcal{T} . This results in a piecewise linear structure on \mathcal{T} . Next, study the level curves of \bar{g} on a 2-dimensional complex which is homotopically equivalent to $\mathcal{C}(\Omega, \partial\Omega, \mathcal{T})$, embedded in \mathbf{R}^3 , obtained by using \bar{g} as a height function on $\mathcal{C}(\Omega, \partial\Omega, \mathcal{T})$. We will work with the level curves of \bar{g} or equivalently, with their projection on $\mathcal{C}(\Omega, \partial\Omega, \mathcal{T})$. The topological structure of the level curves associated with the solutions will be carefully studied. For example, the level curves in the case of an annulus (Theorem 0.4) are simple, piecewise linear closed curves, all of which are in the (free) homotopy class determined by E_1 . One nice consequence of this is that all the rectangles constructed in the proof of Theorem 0.4 are embedded. In the proof of Theorem 0.1, we will show that all the level curves associated to values in $[0, k_1]$, for some constant $k_1 < k$, are simple, piecewise linear closed curves in either the (free) homotopy class of E_2^1 or the class determined by E_2^2 . However, for the value k_1 , it will be proved that the (unique) associated level curve is a figure eight. Furthermore, any level curve of \bar{g} for a value which is larger than k_1 and smaller than or equal to k , is a simple closed curve in the (free) homotopy class determined E_1 . The basic idea of Theorem 0.1 is to cut $(\Omega, \partial\Omega, \mathcal{T})$ along the (projection of) a figure eight curve, tile each cylinder according to Theorem 0.4, and glue back.

We will often work with a series of modified boundary value problems. Each is a slight modification of the initial problem. This important feature of the proofs is due to the fact that the level sets of the original boundary value problem (defined on $\mathcal{T}^{(0)}$) intersect $\mathcal{T}^{(1)}$ in a set which is much larger than $\mathcal{T}^{(0)}$ (see §2 for details).

In fact, once Theorem 0.4 is proved, we proceed to prove Theorem 0.2 by an inductive process. Unlike common proofs in the theory of surfaces, in which a surface is cut along closed geodesics yielding a pair of pants decomposition, we keep cutting our region along particular (projection of) singular level curves until we encounter a planar pair of pants or an annulus. This is a subtle point, arising from the fact that our gluing needs to preserve lengths of curves determined by two boundary value problems (see Definition 1.10). A technical point (which will be addressed in §4) is that one boundary component of one of the pieces (or more) in the decomposition will be singular, hence Theorem 0.4 cannot be applied directly.

The paper is organized as follows. In §1 we introduce notation, recall a few useful facts concerning boundary value problems on graphs, and define a new natural metric induced by a solution of a boundary value problem (see Definition 1.9). In §2 we carry out analysis of level curves of the D-BVP solution; our study brings analysis and topology together in

order to provide a good notion of length for level curves as well as a topological description of them. In §3, we prove Theorem 0.4, and §4 is devoted to the proofs of Theorem 0.1 and Theorem 0.2. Figures 2.10–2.12, Figure 3.1, and Figure 4.16 were generated by two lengthy programs written in Mathematica (version 7.0) by the author and will be available upon request.

Remark 0.5. The assertions of Theorem 0.4 may (in principle) be obtained by employing techniques introduced in the famous paper by Brooks, Smith, Stone and Tutte ([10]), in which they study square tilings of rectangles. They define a correspondence between square tilings of rectangles and planar multigraphs endowed with two poles, a source, and a sink. They view the multigraph as a network of resistors in which current is flowing. In their correspondence, a vertex corresponds to a connected component of the union of the horizontal edges of the squares in the tiling; one edge appears between two such vertices for each square whose horizontal edges lie in the corresponding connected components. Their approach is based on *Kirckhoff's circuit laws* that are widely used in the field of electrical engineering. We found the sketch of the proof of Theorem 0.4 given in [10] hard to follow. In fact, another proof of a slight generalization of this theorem was given by Benjamini and Schramm ([7], see also [8] for a related study) using techniques from probability and the dual graph of \mathcal{T} . It is interesting to recall that it was Dehn [14, 1903] who was the first to show a relation between square tiling and electrical networks. In an elegant paper, combining a mixture of geometry and probability, Kenyon ([23]) used the fact that a resistor network corresponds to a reversible Markov chain. He showed a correspondence between planar non-reversible Markov chains and trapezoid tilings. A completely different method, for the case of tiling a rectangle by squares, was given using *extremal length* arguments in [26] by Schramm. One should also mention that Cannon, Floyd and Parry (see [12]), using extremal length arguments (similar to these in [26]), provide another proof for the existence of tiling by squares. Both [26] and [12] are widely known as “a finite Riemann mapping theorem” and serve as the first step in Cannon’s combinatorial Riemann mapping theorem ([11]). We include our proof of Theorem 0.4, which is guided by similar principles to some of the ones mentioned above, yet significantly different in a few points, in order to make this paper self-contained. In addition, the important work of Bendito, Carmona and Encinas (see for example [4],[5],[6]) on boundary value problems on graphs allows us to use a unified framework to even more general problems. Their work is essential to our applications and we will use parts of it quite frequently in this paper as well as in its sequels ([19],[20]).

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1. PRELIMINARIES - BOUNDARY VALUE PROBLEMS ON GRAPHS

We recall some known facts regarding harmonic functions and boundary value problems on networks. We use the notation of Section 2 in [3]. Let $\Gamma = (V, E, c)$ be a *finite network*, that is a simple and finite connected graph with vertex set V and edge set E . Since in this paper $\Gamma = (V, E, c)$ is induced by $(\Omega, \partial\Omega, \mathcal{T})$, we shall further assume that the graph is planar. Each edge $(x, y) \in E$ is assigned a *conductance* $c(x, y) = c(y, x) > 0$. Let $\mathcal{P}(V)$ denote the set of non-negative functions on V . If $u \in \mathcal{P}(V)$, its support is given by $S(u) = \{x \in V : u(x) \neq 0\}$. Given $F \subset V$ we denote by F^c its complement in V . Set $\mathcal{P}(F) = \{u \in \mathcal{P}(V) : S(u) \subset F\}$. The set $\partial F = \{(x, y) \in E : x \in F, y \in F^c\}$ is called the *edge boundary* of F and the set $\delta F = \{x \in F^c : (x, y) \in E \text{ for some } y \in F\}$ is called the *vertex boundary* of F . Let $\bar{F} = F \cup \delta F$ and let $\bar{E} = \{(x, y) \in E : x \in F\}$. Given $F \subset V$, let $\bar{\Gamma}(F) = (\bar{F}, \bar{E}, \bar{c})$ be the network such that \bar{c} is the restriction of c to \bar{E} . We say that $x \sim y$ if $(x, y) \in \bar{E}$. For $x \in \bar{F}$ let $k(x)$ denote the degree of x (if $x \in \delta(F)$ the neighbors of x are taken only from F).

The following are discrete analogues of classical notions in continuous potential theory [16].

Definition 1.1. ([4, Section 3]) Let $u \in \mathcal{P}(\bar{F})$. Then for $x \in \bar{F}$, the function $\Delta u(x) = \sum_{y \sim x} c(x, y)(u(x) - u(y))$ is called the Laplacian of u at x , (if $x \in \delta(F)$ the neighbors of x are taken only from F) and the number

$$(1.2) \quad E(u) = \sum_{x \in \bar{F}} \Delta u(x)u(x) = \sum_{(x, y) \in \bar{E}} c(x, y)(u(x) - u(y))^2,$$

is called the *Dirichlet energy* of u . A function $u \in \mathcal{P}(\bar{F})$ is called harmonic in $F \subset V$ if $\Delta u(x) = 0$, for all $x \in F$.

For example when $c(x, y) \equiv 1$, u is harmonic at a vertex x if and only if the value of u at x is the arithmetic average of the value of u on the neighbors of x . A fundamental property which we will often use is the *maximum property*, asserting that if u is harmonic on $V' \subset V$, where V is a connected subset of vertices having a connected interior, then u attains its maximum and minimum on the boundary of V' (see [27, Theorem I.35]).

For $x \in \delta(F)$, let $\{y_1, y_2, \dots, y_m\} \in F$ be its neighbors enumerated clockwise. The *normal derivative* (see [13]) of u at a point $x \in \delta(F)$ with respect to a set F is

$$(1.3) \quad \frac{\partial u}{\partial n}(F)(x) = \sum_{y \sim x, y \in F} c(x, y)(u(x) - u(y)).$$

The following proposition establishes a discrete version of the first classical *Green identity*. It plays a crucial role in the proof the main theorem in [18] and is essential in this paper.

Theorem 1.4. ([3, Prop. 3.1]) **(The first Green identity)** *Let $F \subset V$ and $u, v \in \mathcal{P}(\bar{F})$. Then we have that*

$$(1.5) \quad \sum_{(x, y) \in \bar{E}} c(x, y)(u(x) - u(y))(v(x) - v(y)) = \sum_{x \in F} \Delta u(x)v(x) + \sum_{x \in \delta(F)} \frac{\partial u}{\partial n}(F)(x)v(x).$$

Remark 1.6. In [3], a second Green identity is obtained. In this paper we will use only the one above. In [6] (see in particular Section 2 and Section 3), a systematic study of discrete calculus on n -dimensional (uniform) grids of Euclidean n -space is provided. Their definition of a tangent space may be adopted to our setting and does not require the notion of directed edges.

Let \mathcal{T} denote a fixed cellular decomposition of $\Omega \cup \partial\Omega$, and let $F \subset \mathcal{T}^{(0)}$ be given. Let $\{c(x, y)\}_{(x, y) \in \bar{E}}$ be a fixed conductance function, and let $\bar{\Gamma}(F)$ be the associated network. We are interested in functions that solve a boundary value problem (D-BVP) on $\bar{\Gamma}(F)$. The following definition is based on [3, Section 3] and [5, Section 4].

Definition 1.7. Let $k > 0$ be a constant. A solution of a Dirichlet boundary value problem defined on $\bar{\Gamma}(\Omega)$ is a function $f \in \mathcal{P}(\Omega)$ such that f is harmonic in F , $f|_{E_1} = k$ and $f|_{E_2} = 0$, for some positive constant k .

Remark 1.8. The uniqueness and existence of a Dirichlet boundary value solution is guaranteed by the foundational work in [3, Section 3] and [5, Section 4]. In fact, their work provides a detailed framework for a broader class of boundary value problems on finite networks.

A *metric* on a finite network is a function $\rho : V \rightarrow [0, \infty)$. In particular, the length of a path is given by integrating ρ along the path (see [11] and [15] for a different definition). When $\rho \equiv 1$, the familiar distance function on $V \times V$ is obtained by setting $\text{dist}(A, B) = (\sum_{x \in \alpha} 1) - 1 = k$, where $\alpha = (x, x_1, \dots, x_k)$ is a path with the smallest possible number of vertices among all the paths connecting a vertex in A and a vertex in B . We now define a “metric” which will be used throughout this paper.

Definition 1.9. Let $F \subset V$ and let $f \in \mathcal{P}(\bar{F})$. The *flux-gradient metric* is defined by

$$(1.10) \quad \rho(x) = \frac{\partial f}{\partial n}(F)(x), \quad \text{if } x \in \delta(F).$$

This allows us to define a notion of length to any subset of the vertex boundary of F by declaring:

$$(1.11) \quad \text{Length}(\delta F) = \left| \sum_{x \in \delta F} \frac{\partial f}{\partial n}(F)(x) \right|.$$

In the applications of this paper, we will use the second part of the definition in order to define length of connected components of level curves of the D-BVP solution. In [18, Definition 3.3], we defined a similar metric (l_2 -gradient metric) proving several length-energy inequalities.

2. TOPOLOGY AND GEOMETRY OF PIECEWISE LINEAR LEVEL CURVE

Let \mathcal{G} be a polyhedral surface and consider a function $f : \mathcal{G}^{(0)} \rightarrow \mathbf{R} \cup \{0\}$ such that two adjacent vertices are given different values. Let $v \in \mathcal{G}^{(0)}$, and let w_1, w_2, \dots, w_k be its k neighbors enumerated counterclockwise. Following [24, Section 3], consider the number of sign changes in the sequence $\{f(w_1) - f(v), f(w_2) - f(v), \dots, f(w_k) - f(v), f(w_1) - f(v)\}$, which we will denote by $\text{Sgc}_f(v)$. The index of v is defined as

$$(2.1) \quad \text{Ind}_f(v) = 1 - \frac{\text{Sgc}_f(v)}{2}.$$

Definition 2.2. A vertex whose index is different from zero will be called singular; otherwise the vertex is regular. A level set which contains at least one singular vertex will be called singular; otherwise the level set is regular.

A connection between the combinatorics and the topology is provided by the following theorem, which may be considered as a discrete Hopf-Poincaré Theorem.

Theorem 2.3. ([2, Theorem 1], [24, Theorem 2]) (**An index formula**) *We have*

$$(2.4) \quad \sum_{v \in \mathcal{G}} \text{Ind}_f(v) = \chi(\mathcal{G}).$$

Remark 2.5. Due to the topological invariance of $\chi(\mathcal{G})$, note that once the equation above is proved for a triangulated polyhedron, it holds (keeping the same definitions for $\text{Sgc}_f(\cdot)$ and $\text{Ind}_f(\cdot)$) for any cellular decomposition of $\chi(\mathcal{G})$. Also, while the theorem above is stated and proved for a closed polyhedral surface, it is easy to show that it holds in the case of a surface with boundary, where there are no singular vertices on the boundary (simply by doubling along the boundary).

Henceforth, we will keep the notation of Section 0 and Section 1. A key ingredient in our proofs of the theorems stated in the introduction is the ability to define a length for a level curve of g . The main difficulty in defining such a quantity (for level curves) is the fact that these are not piecewise linear curves of the initial cellular decomposition; hence we cannot directly define the weight ρ along them (see Equation (1.10)).

Suppose that L is a fixed, simple, closed level curve and let $\mathcal{O}_1, \mathcal{O}_2$ be the two distinct connected components of L in Ω with L being the boundary of both (this follows by using the Jordan curve theorem; see for instance [9, Theorem III.5.G] for an interesting proof). We will call one of them, say \mathcal{O}_1 , an *interior domain* if all the vertices which belong to it have g -values that are smaller than the g -value of L . The other domain will be called the *exterior domain*. Note that, by the maximum principle, one of $\mathcal{O}_1, \mathcal{O}_2$ must have all of its vertices with g -values smaller than L .

Let $e \in \mathcal{T}^{(1)}$ and $x = e \cap L$. For $x \notin \mathcal{T}^{(0)}$, we have now created two new edges (x, v) and (u, x) . We may assume that $v \in \mathcal{O}_1$ and $u \in \mathcal{O}_2$. We now define conductance constants $\tilde{c}(v, x)$ and $\tilde{c}(x, u)$ by

$$(2.6) \quad \tilde{c}(v, x) = \frac{c(v, u)(g(v) - g(u))}{g(v) - g(x)} \quad \text{and} \quad \tilde{c}(x, u) = \frac{c(v, u)(g(u) - g(v))}{g(u) - g(x)}.$$

We repeat the process above for any new vertex formed by the intersection of L with $\mathcal{T}^{(1)}$. By adding all the new vertices and edges, as well as the piecewise arcs of L determined by the new vertices, we obtain two cellular decompositions, $\mathcal{T}_{\mathcal{O}_1}$ of \mathcal{O}_1 and $\mathcal{T}_{\mathcal{O}_2}$ of \mathcal{O}_2 . Also, two conductance functions are now defined on the one-skeleton of these cellular decompositions by modifying the conductance function for g according to Equation (2.6) (i.e. changes are occurring only on new edges). We will denote these by $\mathcal{C}_{\mathcal{O}_1}$ and $\mathcal{C}_{\mathcal{O}_2}$ respectively (on the arcs of L the conductance is identically zero).

Definition 2.7. For $i = 1, 2$, we let g_i denote the solution of the D-BVP determined on the induced cellular decomposition on \mathcal{O}_i defined by the following:

- (1) for any boundary component E of \mathcal{O}_i and for every vertex $x \in E$, $g_i(x) = g(x)$,
- (2) the conductance function on \mathcal{O}_i is $\mathcal{C}_{\mathcal{O}_i}$, and
- (3) g_i is harmonic in \mathcal{O}_i .

It follows from Equation (2.6), Definition 1.1, and the existence and uniqueness theorems in [3] that for $i = 1, 2$, g_i exists and that $g_i = g|_{\mathcal{O}_i}$. We may now define a flux-gradient metric for L by using Equations (1.10) and (1.11). However, unlike the situation with the boundary components of Ω , we have two possible choices, i.e. computing the normal derivative along L with respect to \mathcal{O}_1 or with respect to \mathcal{O}_2 . Since in the applications we will cut Ω along a particular L and wish to glue the resulting pieces together along L , these lengths computed with respect to the flux-gradient metric in each domain should be the same. The situation when L is not simple is more complicated and will be addressed after a detailed analysis of the topological structure of level curves has been carried out (see the discussion after Remark 2.19). This analysis is the main core of this section. The next lemma shows that all level curves are in fact closed.

Lemma 2.8. *A level curve for the function g is piecewise linear and closed, and each simple cycle of L contains at least one boundary component of $\partial\Omega$.*

Proof. The assertions of the lemma are certainly true for the components of $\partial\Omega$. Assume now that L is not one of these level curves and furthermore that it is not closed. Let q be a boundary point of L . Since g is extended linearly on edges and in an affine fashion on triangles and quadrilaterals, q may (a priori) be an interior point of an edge of the cellular decomposition, a vertex, or in the interior of a cell. All of these cases are easily ruled out. It remains to prove that any such level curve contains at least one boundary component of $\partial\Omega$. If this is not the case, L (being a finite, closed, polygonal planar line) bounds a union of 2-cells in Ω . This is a violation of the maximum principle for the harmonic function g_1 constructed as in Definition 2.7 (\mathcal{O}_1 being one connected component of the union of the 2-cells bounded by L). \square

Remark 2.9. Since g is extended in an affine fashion along edges, it is clear that two disjoint level curves corresponding to the same g value are at combinatorial distance which is at least one.

The structure of simple curves in the plane can be quite complicated and in fact is not completely understood. In our applications, we need only analyze the topological structure of closed curves arising as level curves of the affine extension (over triangles and quadrilaterals) of a harmonic function as defined above. We will henceforth work in the piecewise category.

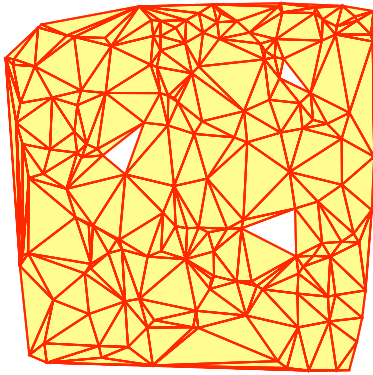


FIGURE 2.10. 4-connected triangulated domain

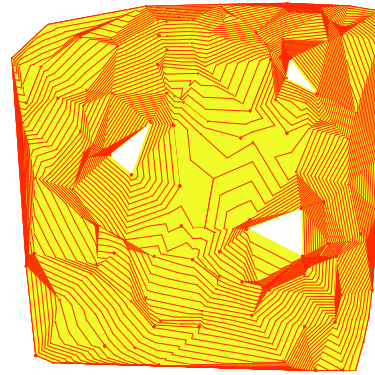


FIGURE 2.11. Level curves for the D-BVP

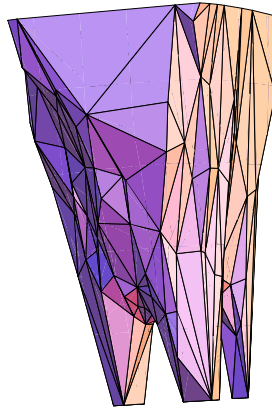


FIGURE 2.12. The surface obtained by using g as a height function.

Definition 2.13. A generalized piecewise bouquet of circles will denote a union of piecewise bouquets of circles where the intersection of any two bouquets is at most a vertex. Moreover, all such tangencies are exterior, i.e., no circle is contained in the interior of the bounded component of another.

Proposition 2.14. *Two simple cycles which correspond to the same g value either are disjoint or intersect at a single vertex. Furthermore, under these assumptions, a simple cycle is never contained in the annulus bounded by a different one (i.e. a “tangency” is always external).*

Proof. Let L_1 and L_2 be two given simple cycles corresponding to the g -value l . Let A_1 and A_2 be the two (piecewise) annuli which they bound respectively. Assume first that $A_1 \cap A_2 = \emptyset$ and that $E = L_1 \cap L_2$ is a piecewise arc. Let I and J be the endpoints of E . As mentioned before (see Lemma 2.8), both I and J are vertices. The link of the vertex I must contain a two cell (a triangle or a quadrilateral) such that an edge between two of its vertices crosses E . This implies that one of these vertices has g -value which is larger than l and the other has g -value which is smaller than l . This is absurd, since by the maximum principle all the vertices in $A_1 \cup A_2$ which do not belong to $g^{-1}(l)$ have g -values that are

strictly smaller than l (none of the vertices of these cells belong to $g^{-1}(l)$). It may also happen that $L_1 \cap L_2 = \{P_1, P_2, \dots, P_k\}$, where $k \geq 1$ is an integer, and where the points P_i are oriented clockwise. Let t_2 be the arc connecting P_1 to P_2 which belongs to L_2 and which does not pass through any other of the P_i 's. Let t_1 be the arc in L_1 which connects P_1 to P_2 which does not pass through any of the P_i 's. Then $t_1 \cup t_2$ is a simple closed level curve which bounds an annulus (see Lemma 2.8). The link of the vertex P_1 must contain a triangle or a quadrilateral with one of its edges crossing t_1 . As above, this leads to a violation of the maximum principle.

Assume now that $A_1 \cap A_2 \neq \emptyset$ and (without loss of generality) that $A_1 \cup L_1$ contains an arc t of L_2 . Let its endpoints be P and Q (they both lie on L_1). Then $L_1 \cup ((L_2 \setminus t) \cup E)$ consists of two simple cycles that satisfy all the conditions described in the case above; hence, this case may not occur.

We now consider the case in which one of the annuli is contained in the other. Without loss of generality, assume that $A_1 \subset A_2$ and that $E = L_1 \cap L_2$ is a piecewise arc. Let I and J be the endpoints of E . The link of the vertex I must contain a two-cell, included in A_2 , where one of its edges connects a vertex in A_1 to a vertex in $A_2 \setminus (A_1 \cup L_1)$. This edge crosses L_1 ; hence one of these vertices has g -value which is larger than l and the other has g -value which is smaller than l ; this is absurd, since by the maximum principle all the vertices in A_2 which do not belong to $g^{-1}(l)$ must have their g value strictly smaller than l . It may also be the case that $L_1 \cap L_2 = \{P_1, P_2, \dots, P_m\}$ where $m \geq 1$ is an integer and the points P_i are oriented clockwise. Note that we allow all the P_i 's to be a single vertex. Consideration of the link of any of the P_i 's, and an argument similar to the one above, leads again to a violation of the maximum principle.

Finally, assume that $L_1 \cap L_2 = \emptyset$. Observe that $A_2 \setminus A_1$ is an annulus; both of its boundary curves have g value l . By a construction analogous to the one described before Definition 2.7, we obtain a new BVP problem defined on a cellular decomposition of $(A_2 \setminus A_1) \cup L_1 \cup L_2$. Its solution is harmonic on $A_2 \setminus A_1$ and has the same value on the boundary components; hence it is the constant function. The cellular decomposition of the annulus contains edges (part of edges) or vertices (and perhaps both) of the initial cellular decomposition. Moreover, the values of the solution and g coincide on $(A_2 \setminus A_1) \cup L_1 \cup L_2$. Both cases lead to the conclusion that connected pieces of edges of the initial cellular decomposition in the annulus have the same g -value, which is absurd. □

Theorem 2.15. *Let L be a connected level curve for g . Then each connected component of L is a generalized piecewise bouquet of circles.*

Proof. First, we only use the fact that L is a closed polygonal line. Let \mathcal{S} be the collection of all the self intersection points of L . By definition, edges of the polygonal (closed) curve L may cross only at points from \mathcal{S} . If L is simple, the assertion follows immediately. Assume that L is not simple. It is well known that we may express L as a union of simple close polygonal curves. The intersection of any two cycles in this decomposition is a union of vertices and edges. Proposition 2.14 forces restrictions on any such decomposition. In particular, any two simple cycles in such a decomposition are either disjoint or intersect in a single vertex, none of which is contained in the annuli bounded by the other. The assertion of the theorem follows immediately. □

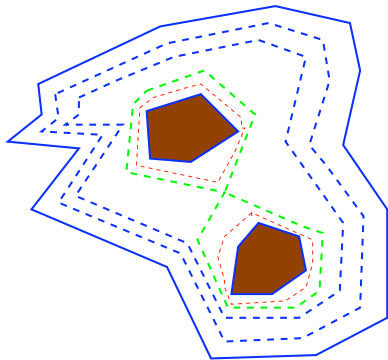


FIGURE 2.16. Level curves in a pair of pants

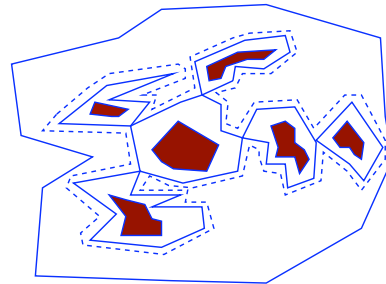


FIGURE 2.17. Level curves in a 7 connected domain

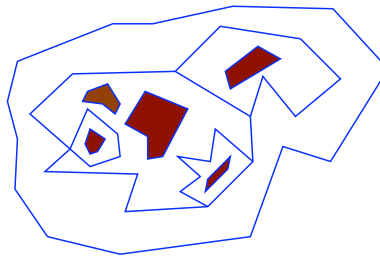


FIGURE 2.18. A non-possible level curve in a 6 connected domain

Remark 2.19. Our interest here is in the existence of a decomposition of the level curve into simple pieces. One may also consider the question, or rather a series of questions, regarding a decomposition of the non-simple polygon determined by a general closed polygonal curve. For many deep questions and an excellent survey on this interesting subject (some of the questions involved are known to be *NP*), see for instance [22].

The topological structure of a general connected component of a level curve L (Theorem 2.15) means that the complement of L in $\Omega \cup \partial\Omega$ is composed of a bounded domain (which is a disjoint union of annuli) in which all vertices have g -values smaller than the g -value of L , and a second domain whose boundary consists of L and possibly other curves in $\partial\Omega$, where L is in the boundary of both. Following a construction similar to the one preceding Definition 2.7, we may now define two modified conductance functions and two solutions of D-BVP problems. By abuse of notation, we call the first domain \mathcal{O}_1 , the second \mathcal{O}_2 , and we let $\mathcal{C}_{\mathcal{O}_1}, \mathcal{C}_{\mathcal{O}_2}, g_1, g_2$ be the analogous quantities. The corresponding two new cellular decompositions and their union will henceforth be called the *induced* cellular decomposition.

Theorem 2.20. *Let L be a level curve for g . Then the following equality holds*

$$(2.21) \quad l_{\mathcal{O}_1}(L) = l_{\mathcal{O}_2}(L),$$

where the left-hand side denotes the length of L measured with respect to the flux-gradient metric induced by g_1 , and the right-hand side denotes the length of L measured with respect to the flux-gradient metric induced by g_2 .

Proof. First, let us assume that L is simple and corresponds to the g -value h . Since L is at combinatorial distance at least one from any other level curves of g of the same value, the 1-combinatorial neighborhood of L in \mathcal{O}_2 is comprised of vertices that are of g -values greater than h .

Let $x \in \mathcal{T}^{(0)} \cap L$. Let y_1, y_2 be its neighbors in L , $\{x_1, \dots, x_m\}$ in \mathcal{O}_1 and $\{z_1, \dots, z_k\}$ in \mathcal{O}_2 . Since g is harmonic at x we have

$$(2.22) \quad 0 = \Delta g(x) = \sum_{y \sim x} c(x, y) (g(x) - g(y)).$$

Hence, (since $g(x) = g(y_1) = g(y_2)$)

$$(2.23) \quad 0 = \sum_{i=1}^m c(x, x_i) (g(x) - g(x_i)) + \sum_{j=1}^k c(x, z_j) (g(x) - g(z_j)).$$

Let $x \in \mathcal{T}_{\mathcal{O}_1} \cup \mathcal{T}_{\mathcal{O}_2}$ be a new vertex in L (that is $x \notin \mathcal{T}^{(0)}$) with $v \in \mathcal{O}_1$ and $u \in \mathcal{O}_2$. We have

$$(2.24) \quad \frac{\partial g_1}{\partial n}(\mathcal{O}_1)(x) = \tilde{c}(v, x)(g(x) - g(v)) = c(u, v)(g(u) - g(v)),$$

and

$$(2.25) \quad \frac{\partial g_2}{\partial n}(\mathcal{O}_2)(x) = \tilde{c}(u, x)(g(x) - g(u)) = c(u, v)(g(v) - g(u)).$$

By summing the three equations above over all vertices in L , the assertion follows.

By combining the arguments above with the topological structure of a general level curve provided by Theorem 2.15 and a simple modification at singular vertices along L , we will now treat the second case. Let us assume that L is not simple. Recall that a level curve may intersect itself only at a vertex. Let $\mathcal{S} = \{v_1, \dots, v_m\}$ be all such intersection points and $\{\text{Ind}_f(v_1), \dots, \text{Ind}_f(v_m)\}$ be their indices respectively. For a vertex $x \in L$ (of the initial triangulation or the induced one) which is not in \mathcal{S} (note that $\text{Ind}_f(x) = 0$), the arguments above leading to Equation 2.22 go through precisely in the same manner and yield (with $\mathcal{O}_1, \mathcal{O}_2$ modified as explained immediately after Remark 2.19) the same conclusion.

Let $v \in \mathcal{S}$. Then v is in the intersection of C_1, \dots, C_l piecewise simple circles where, $l = \text{Sgc}_g(v)/2$ circles. Let $\{x_1^i, \dots, x_{k(i)}^i\}$ be the neighbors of v in $\mathcal{O}_1 \cap \text{interior}(C_i)$, $\{y_1, \dots, y_m\}$ the neighbors of v in L and $\{z_1, \dots, z_p\}$ in \mathcal{O}_2 . Since g is harmonic at v , we may now modify slightly Equation (2.23). Since $g(v) = g(y_1) = \dots = g(y_m)$, we obtain

$$(2.26) \quad 0 = \sum_{j=1}^l \sum_{i=1}^{k(j)} c(v, x_i^j) (g(v) - g(x_i^j)) + \sum_{j=1}^p c(v, z_j) (g(v) - g(z_j)).$$

The assertion of the theorem now follows by summing over all the vertices in L . Finally, if L has several connected components, the assertion of the theorem follows by addition over each component (which must be at combinatorial distance greater than one from each other). \square

Remark 2.27. We now wish to start using Theorem 2.3 and Remark 2.5. One should note that since boundary components are at the same g -level, a modification is needed. This is

done in the following way. Along each boundary component, add a piecewise linear curve homotopic to it, and at combinatorial distance which is smaller than one. This can be done in such a way that there is exactly one sign change at each new vertex (not taking into account vertices on the boundary). Let $\tilde{\Omega} \in \Omega$ be the region bounded by these curves. Doubling along the boundary of $\tilde{\Omega}$ gives a surface without boundary with all of its singular vertices identical to those of g , for which the above mentioned theorem may be applied.

We end this section with a useful property of level curves. This property is essential in the proofs of Theorem 0.1 and Theorem 0.2. In the course of these proofs, we will need to know that there is a singular level curve which encloses all of the interior components of $\partial\Omega$. This level curve (which is necessarily singular) is the one along which we will cut the surface. In the inductive process, we keep cutting along such level curves over domains of fewer boundary components until the remaining pieces are annuli.

Proposition 2.28. *With notation as above, there exists a singular level curve which contains in the interior of the domain it bounds, all of the inner boundary components of $\partial\Omega$. Such a level curve is unique.*

Proof. We argue by contradiction. Throughout the proof, we use the topological structure of level curves provided by Theorem 2.15. Suppose that the first assertion of this proposition is false. Then every singular level curve omits at least one inner boundary component. Let L be such a curve. Suppose that L omits $k \geq 1$ boundary components and contains $l \geq 2$ boundary components in its interior, where $k+l = m-1$. Let $\{v_1, v_2, \dots, v_n\}$ be the singular vertices that belong to L ; let $\{w_1, w_2, \dots, w_m\}$ be the singular vertices in the interior of L . It is easy to see that (for example by induction)

$$(2.29) \quad \sum_{x \in \{w_1, \dots, w_m\}} \text{Ind}_g(x) = 1 - l.$$

By Theorem 2.3 we must have additional singular vertices $\{q_1, \dots, q_s\}$ such that

$$(2.30) \quad \sum_{y \in \{q_1, \dots, q_s\}} \text{Ind}_g(y) = (1 - k) - 1.$$

Let L_1 be a singular level curve that has q_1 as a singular vertex. Suppose that L_1 does not enclose L in its interior. Then

$$(2.31) \quad \text{Ind}_g(q_1) \leq (1 - k),$$

with equality if and only if L_1 encloses all k boundary components. Therefore, we must have $s = 2$, and a singular level curve that passes through q_2 will necessarily enclose all interior components of $\partial\Omega$. Hence, we arrive at a contradiction.

Assume now that L_1 encloses L and $t \geq 1$ out of the k boundary components. Let $\{q_1, q_2, \dots, q_s\}$ be the singular vertices on the various cycles on L_1 which do not belong to the cycle containing L or its interior. It is easy to see that

$$(2.32) \quad \sum_{z \in \{q_1, q_2, \dots, q_s\}} \text{Ind}_g(z) = (1 - t).$$

By Theorem 2.3 we must have additional singular vertices $\{p_1, \dots, p_n\}$ such that

$$(2.33) \quad \sum_{t \in \{p_1, \dots, q_n\}} \text{Ind}_g(t) = -1 - (k - t).$$

We now repeat the process above with L_1 replacing L , $l + t$ replacing l and $k - t$ replacing k . After finitely many times (at most $k - 1$), we must end up with a singular level curve with a single singular vertex of index -2 that encloses all inner boundary components of $\partial\Omega$. \square

We are now concerned with geometric properties of the set of lengths of level curves. This is significant for the applications. For example, even in the case of an annulus (which is the subject of Theorem 0.4), it is not clear that level curves in the same homotopy class have the same length. Since we wish to map a given annulus to a straight cylinder, we must verify this property and its suitable generalizations.

Theorem 2.34. *There exists a finite set of non-negative numbers \mathcal{K} such that the following holds:*

- (1) \mathcal{K} contains 0, and k is its maximal element,
- (2) \mathcal{K} is monotone decreasing,
- (3) any level curve corresponding to a g -value which does not belong to \mathcal{K} is simple, and
- (4) any component of a level curve, which corresponds to a g -value which is strictly between any two values $k_n > k_m$ in \mathcal{K} , such that $k_n - k_m$ is minimal, and is contained in a unique simple cycle C_n determined by k_n , has its l_1 -length equal to that of C_n .
- (5) Moreover, the length of C_n is equal to the length of the component of $g^{-1}(k_m)$ which it encloses.

Proof. We let \mathcal{K} denote the set of critical values of g union k and 0; this gives assertion (1). Since $\chi(\Omega \cup \partial\Omega)$ is negative and the index of each singular vertex is less than or equal to -1 , Theorem 2.3 implies that the number of singular vertices, and hence the number of corresponding critical g -values, is finite. Assertion (2) follows by ordering. Assertion (3) is the content of Lemma 2.8. We now turn to proving (4). Let L be a connected component of a level curve as described in (4). By the structure theory provided in Theorem 2.15 and the maximum principle, L is contained in a unique simple cycle, part of the bouquet which composes $g^{-1}(k_m)$. Let C_n denote this cycle. We first assume that the combinatorial distance between L and C_n is greater than or equal to 2. We apply the construction that precedes Definition 2.7 to the level curves C_n and L and the annulus \mathcal{A}_{L, C_n} enclosed by them. By abuse of notation, we will keep denoting the modified set of vertices by $\mathcal{T}^{(0)}$.

Let $F = F_{C_n, L}$ denote the subset of $\mathcal{T}^{(0)} \cap \mathcal{A}_{L, C_n}$. Since the combinatorial distance between C_n and L is greater than or equal to 2, it follows that

$$(2.35) \quad F \neq \emptyset \text{ and that } \delta(F) = \{v \in \mathcal{T}^{(0)} : v \in C_n \cup L\}.$$

Recall that g_1 , the modified solution of the D-BVP, is defined on $F \cup \delta(F)$ by requiring that g_1 is harmonic (with respect to the modified conductance function) in F , $g_1|_L = g|_L$ and $g_1|_{C_n} = g|_{C_n}$. By the uniqueness of the D-BVP solution, it is clear that $g_1 = g|_F$. Note that

$F^\circ = F$ is the set of vertices in F which do not belong to $\delta(F)$. We apply Proposition 1.4 (the first Green Identity) to F with $u = g_1$ and $v \equiv 1$. Equation (1.5) then gives

$$(2.36) \quad 0 = \sum_{x \in \delta(F)} \frac{\partial g_1}{\partial n}(F^\circ)(x).$$

Hence

$$(2.37) \quad 0 = \sum_{x \in C_n} \frac{\partial g_1}{\partial n}(F^\circ)(x) + \sum_{x \in L} \frac{\partial g_1}{\partial n}(F^\circ)(x),$$

which implies that

$$(2.38) \quad \sum_{x \in L} \frac{\partial g_1}{\partial n}(F^\circ)(x) = - \sum_{x \in C_n} \frac{\partial g_1}{\partial n}(F^\circ)(x).$$

It follows that, with respect to the flux-gradient metric (see Equation (1.11)), L and C_n have the same length. We now deal with the case that the combinatorial distance between C_n and L is less than 2. While we wish to use again the first Green Identity, care is needed since F is empty in this case. We add a vertex on each edge in the modified cellular decomposition that has a vertex v on L and a vertex u on C_n . The value of $g_1(t)$ is defined by the value of g_1 on this edge and we also let

$$(2.39) \quad \tilde{c}(v, t) = \frac{\tilde{c}(v, t)(g_1(v) - g_1(u))}{g_1(v) - g_1(t)} \quad \text{and} \quad \tilde{c}(u, t) = \frac{\tilde{c}(u, t)(g_1(u) - g_1(v))}{g_1(u) - g_1(t)},$$

where $\tilde{c}(v, t)$ and $\tilde{c}(u, t)$ are defined as in equation (2.6) with t replacing x . We keep all conductance functions on other edges unchanged.

By applying again the existence and uniqueness theorems in [3], we obtain a solution g_2 of a new D-BVP defined on \mathcal{A}_{L, C_n} by requiring that $g_2|_L = g_1|_L$, $g_2|_{C_n} = g_1|_{C_n}$ and that g_2 is harmonic in \mathcal{A}_{L, C_n} . It follows that g_2 is g_1 on all vertices in $\mathcal{T}^{(0)}$ and is modified so as to have the values of g_1 on all vertices defined above. It is easy to verify (using Equation 2.6 and Equation 2.39) that

$$(2.40) \quad \frac{\partial g_2}{\partial n}(\mathcal{A}_{L, C_n})(v) = \frac{\partial g_1}{\partial n}(\mathcal{A}_{L, C_n})(v) \quad \text{and} \quad \frac{\partial g_2}{\partial n}(\mathcal{A}_{L, C_n})(u) = \frac{\partial g_1}{\partial n}(\mathcal{A}_{L, C_n})(u).$$

Assertion (5) is proved by following the same techniques that were used in proving assertion (4) and Theorem 2.15. □

Remark 2.41. Observe that once assertion (5) is proved, in order to prove (4), one needs only add at most one vertex on each edge whose vertices lie on C_n and $g^{-1}(k_m)$ respectively. Also, the same techniques employed in the proof of assertion (4) allow one to get equality between the sum of the l_1 -lengths of all the components for a non-singular level curve enclosed in a simple cycle and the l_1 -length of this cycle. Finally, the case $g^{-1}(0)$ was not stated separately, yet it similarly gives that the sum of the l_1 -lengths of the inner boundary components of $\partial\Omega$ equals the l_1 -length of the outer boundary. We will revisit this last point in the proof of Theorem 0.1.

3. THE CASE OF AN ANNULUS

In this section we prove Theorem 0.4. The proof consists of two parts. First, we will show that there is a well-defined mapping from $\mathcal{T}^{(1)}$ into a set of (Euclidean) rectangles embedded in the cylinder $S_{\mathcal{A}}$. The crux of this part is the fact that level curves for g have the same induced length (measured with the l_1 metric, see Remark 2.41) and a simple application of the maximum principle. Second, we will show that the collection of these rectangles forms a tiling of $S_{\mathcal{A}}$ with no gaps and no overlaps. This is a consequence of the first part and an energy-area computation. The latter follows by our construction, the dimensions of $S_{\mathcal{A}}$, and the first Green identity (see Theorem 1.4).

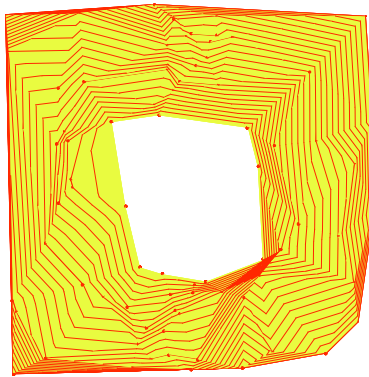


FIGURE 3.1. Level curves in an annulus

Given a straight Euclidean cylinder of height h , we will endow it with coordinates arising naturally from $\mathbf{S}^1 \times [0, h]$. The boundary component corresponding to h will be called the top and the other one will be called the bottom. Before providing the proof, we need a definition which will simplify keeping track of the mapping f .

Definition 3.2. A marker on a straight Euclidean cylinder is a vertical closed interval which is the isometric image of $\theta \times [a, b]$, for some $\theta \in [0, 2\pi)$ and $[a, b] \subset [0, h]$ with $a < b$. The marker's uppermost end-point corresponds to (θ, b) and its lowest end-point to (θ, a) .

Proof of Theorem 0.4. Let $S_{\mathcal{A}}$ be a straight Euclidean cylinder with height $H = k$ and circumference

$$C = \sum_{x \in E_1} \frac{\partial g}{\partial n}(x).$$

Let $\mathcal{L} = \{L_1, \dots, L_k\}$ be the level sets for g corresponding to the vertices in $\mathcal{T}^{(0)}$ arranged in a descending g -values order. We place a vertex at each intersection of an edge with an L_i , $i = 1, \dots, k$, and if necessary more vertices on edges so that any two successive level curves in \mathcal{L} are at combinatorial distance (at least) two. We call the first group of added vertices type I and the second type II (recall that conductance along edges are changed as well, according to the discussion preceding Definition 2.7). In particular, the assertions of Theorem 2.34 and Remark 2.41 hold. Since $\chi(\mathcal{A}) = 0$ and the index of a singular vertex is always negative, the index formula (Theorem 2.3) prevents the existence of singular vertices in \mathcal{A} . Therefore, $\mathcal{K} = \{0, k\}$, and furthermore the length of any g -level curve is equal to C .

Let the vertices $\{x_1, \dots, x_p\}$ in $E_1 = L_1$ be ordered counterclockwise (and on any level curve), starting with x_1 . Let $\{y_1, y_2, \dots, y_t\}$ be its type I neighbors in the induced cellular decomposition oriented clockwise (which will always be the ordering for neighbors). We identify x_1 with $0 \times k$ in the coordinates above. We associate markers $\{m_{x_1, y_1}, \dots, m_{x_1, y_t}\}$ with x_1 in the following way. The length of marker m_{x_1, y_s} , for $s = 1, \dots, t$, is equal to (the constant) $g(x_1) - g(y_s)$ and its uppermost end-point is positioned at the top of S_A at

$$(3.3) \quad x_1 + \frac{\sum_{k=1}^{s-1} c(x_1, y_k)(g(x_1) - g(y_k))}{\frac{\partial g}{\partial n}(\mathcal{A})(x_1)} \times 2\pi,$$

measured counterclockwise. For each edge $e_{u,v} = [u, v]$ with $g(u) > g(v)$, let $Q_{u,v}$ be a Euclidean rectangle with height equals to $g(u) - g(v)$ and width equals to $c(u, v)(g(u) - g(v))$. We will identify a planar straight Euclidean rectangle and its image, under an isometry, in a straight Euclidean cylinder. For $s = 1, \dots, t$, we position Q_{x_1, y_s} in S_A in such a way that its leftmost edge coincides with m_{x_1, y_s} . By construction and the position of the markers,

$$(3.4) \quad Q_{x_1, y_s} \cap Q_{x_1, y_{s+1}} = m_{x_1, y_{s+1}}.$$

Assume that we have placed markers and rectangles associated to all the vertices up to x_k where $k < p$; let z_1 be the leftmost neighbor of x_{k+1} and let $Q_{x_k, v}$ be the rightmost rectangle associated with x_k . We now position the marker m_{x_{k+1}, z_1} , associated with x_{k+1} and z_1 so that it is lined with the rightmost edge of $Q_{x_k, v}$ and his upper end-point is at the top of S_A . We continue placing markers and rectangles corresponding to the rest of the neighbors of x_{k+1} . We terminate these steps when $k = p$. Note that the top of S_A is completely covered by the top of the rectangles constructed above where intersections between any of these (top edges) is either a vertex or empty.

For all $1 < n < k$, assume that all the markers corresponding to vertices in L_{n-1} and their associated rectangles have been placed as above in such a way that the following condition, which we call *consistent*, holds. For $[w, v] \in \mathcal{T}^{(1)}$ with $g(w) > g(v)$ and $s \in [w, v]$ a vertex of type I, the uppermost end-point of the marker $m_{s,v}$ coincides with the lowest end-point of the marker $m_{w,s}$; moreover, the two rectangles $Q_{w,s}, Q_{s,v}$ tile $Q_{v,w}$. Informally, this condition allows us to “continuously extend” rectangles associated with edges that cross level curves along these, and therefore will show that edges in $\mathcal{T}^{(1)}$ are mapped (perhaps in several steps) onto a unique rectangle.

We will now show how to place the markers and rectangles corresponding to the vertices of the level set L_n in a consistent way. The uppermost end-point of each marker associated with a vertex $v \in L_n$ in this level set, and any of its neighbors in L_{n+1} is placed in S_A at height $g(v)$. Observe that v is a vertex in some $[q_i, v]$, where q_i belongs to L_n . Choose among all such edges the rightmost (viewed from L_n). Let $[q_0, v]$ be this edge and let $m_{q_0, v}$ be its marker. Place the marker of v which corresponds to an edge $[v, w]$ with w being the leftmost vertex among the neighbors of v in L_{n+1} , so that its uppermost end-point coincides with the lowest end-point of $m_{q_0, v}$. Let s_0 be a vertex of type I on L_n . By definition, s_0 is connected to a unique vertex $v_0 \in L_{n+1}$ and to a unique vertex $w_0 \in L_{n-1}$. Let s_q be the first vertex to the left of s_0 of type I in L_n . By definition, s_q is connected to a unique vertex w_p in L_{n-1} and to a unique vertex $v_l \in L_{n+1}$. Let $\{s_1, \dots, s_{q-1}\}$ be the vertices on L_n

between s_0 and s_q and let $\{w_1, \dots, w_{p-1}\}$ be the vertices on L_{n-1} between w_0 and w_p . Let $\mathcal{Q}_1 = \mathcal{Q}_{w_0, s_0, s_q, w_p}$ be the piecewise linear rectangle enclosed by $[w_0, s_0] \cup [w_p, s_q] \cup L_{n-1} \cup L_n$, and which contains $\{s_0, \dots, s_q\}$, and let $\mathcal{Q}_2 = \mathcal{Q}_{s_0, v_0, s_q, v_l}$ be the piecewise linear rectangle enclosed by $[s_0, v_0] \cup [s_q, v_l] \cup L_n \cup L_n$, and which contains $\{s_0, \dots, s_q\}$.

In order to prove that the consistent condition holds for all markers and rectangles created in this step, it suffices to prove it at s_q , assuming (with out loss of generality) that the first marker we placed in a consistent way is m_{s_0, v_0} . By construction (see in particular Equation (3.3)) we need to prove that

$$(3.5) \quad \sum_{i=1}^{p-1} \frac{\partial g}{\partial n}(\mathcal{Q}_1)(w_i) + \frac{\partial g}{\partial n}(\mathcal{Q}_1)_{\text{right}}(w_0) = \sum_{i=1}^{q-1} \frac{\partial g}{\partial n}(\mathcal{Q}_2)(s_i) + \frac{\partial g}{\partial n}(\mathcal{Q}_2)_{\text{right}}(s_0),$$

where the subscript “right” indicates that neighbors in the expressions are taken from \mathcal{Q}_1 (first line) or \mathcal{Q}_2 (second line) only. It is easy to check that since the modification of g is harmonic at each $s_i, i = 0, \dots, q$, and since s_0 and s_q are type I vertices, Equation (3.5) holds.

To conclude the construction, continue as above, exhausting all vertices in L_n . By the maximum principle, our construction, and the fact that all level curves have their flux-gradient length equal to C , it is clear that the union of the rectangles is contained in $S_{\mathcal{A}}$.

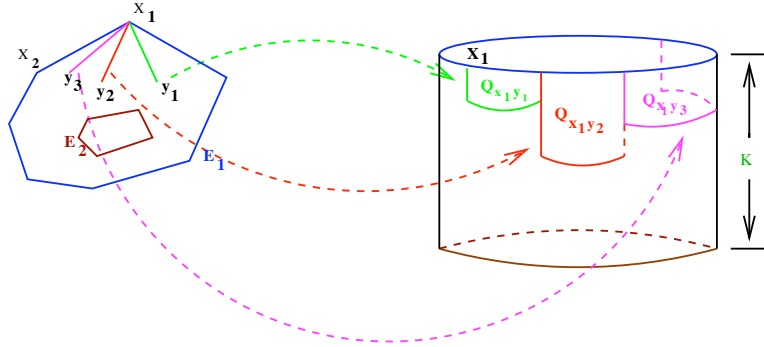


FIGURE 3.6. Several markers and rectangles on the Euclidean cylinder after the completion of the construction.

We now prove that the collection of rectangles constructed above tiles $S_{\mathcal{A}}$ leaving no gaps. Without loss of generality, suppose that the collection of rectangles does not cover a strip of the form $[\theta_1, \theta_2] \times [h_0, h_1]$ in $S_{\mathcal{A}}$, where $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $0 \leq h_0 < h_1 \leq k$. By harmonicity, there exists at least one path whose vertices belong to $\mathcal{T}^{(0)}$ such that the values of g along this path are strictly decreasing from k to 0 .

In particular, the value h_1 is attained on some edge or vertex of this path. By construction, a gap in a g -level curve (i.e. an arc of level curve which is not covered by the top edges of rectangles) never occurs when h_1 is the g -value associated to a vertex in the modified cellular decomposition. Hence, we may assume that h_1 is attained in the interior of an edge. Let L_{h_1} be the corresponding level curve. Recall that L_{h_1} is simple and closed (as are all other level curves of g ; see Lemma 2.8). We now follow the construction preceding Definition 2.7

and let $\{u_1, u_2, \dots, u_q\}$ be all the new vertices on L_{h_1} , that is we place a vertex at each intersection of an edge in $\tilde{\mathcal{T}}^{(1)}$ with L_{h_1} . As mentioned in the first paragraph of the proof, the flux-gradient length of L_{h_1} is equal to C . Moreover, this length is equal to

$$(3.7) \quad \sum_{i=1}^q \frac{\partial g}{\partial n}(\mathcal{O}_1)(u_i),$$

where \mathcal{O}_1 is the interior of the annulus enclosed by L_{h_1} and E_2 (see the discussion proceeding Definition 2.7). In particular, we may now place markers and rectangles associated to the collection of vertices $\{u_1, u_2, \dots, u_q\}$ so that L_{h_1} is covered by the top edges of these rectangles. Recall that up to this moment, markers and rectangles were associated only to vertices of type I in addition to those in $\mathcal{T}^{(0)}$. It is very easy to check that this can be done in a consistent way (using Equation (2.6)). Since g is extended affinely over edges, every value between h_0 and h_1 is attained by g . Repeating this argument shows that all level curves are covered by rectangles. Hence the collection of rectangles leaves no gaps in $S_{\mathcal{A}}$.

Using an area argument, we now finish the proof by showing that there is no overlap between any two of the rectangles. Let \mathcal{R} be the union of all the rectangles. By definition,

$$(3.8) \quad \text{Area}(\mathcal{R}) = \sum_{[x,y] \in \tilde{\mathcal{T}}^{(1)}} c(x,y)(g(x) - g(y))(g(x) - g(y)).$$

Note that the sum appearing in the right-hand side of Equation (3.8) is computed over the induced cellular decomposition. A simple computation (using again Equation (2.6)) and the fact that the construction is consistent shows that this sum is equal to the one taken over $[x,y] \in \mathcal{T}^{(1)}$. Hence, the right-hand side of this equation is the energy $E(g)$ (see Definition 1.1). Therefore, by the first Green identity, applied with $u = v = g$ (see Theorem 1.4), and the dimensions of $S_{\mathcal{A}}$, we have

$$(3.9) \quad E(g) = \text{Area}(\mathcal{R}) = \text{Area}(S_{\mathcal{A}}).$$

Hence, since the rectangles do not overlap, they must tile $S_{\mathcal{A}}$. It is also evident that the mapping f constructed above is energy preserving. □

4. THE CASE OF AN m -CONNECTED BOUNDED PLANAR REGION, $m > 2$

There is a technical issue which we need to address in the proofs of Theorem 0.2 and Theorem 0.1. Since Proposition 1.4 is applied all along, we must make sure for example, that the combinatorial distance between any two distinct level curves is at least two. Therefore, one adds as many vertices of type I and type II as needed, and modify conductances accordingly (as done at the beginning of the proof of Theorem 0.4) until the assertions of Theorem 2.34 and Remark 2.41 hold.

4.1. Pair of pants. In this subsection we provide a proof of Theorem 0.1. One interesting ingredient of the proof is the construction of a good mapping (in the sense of Theorem 0.4) from a planar annulus with one singular boundary component into a Euclidean cylinder having one boundary component, a singular curve and the other one which is simple. The construction of the mapping f is then achieved by gluing two Euclidean cylinders to the one above along the singular component.

Proof of Theorem 0.1.

We extend the solution g to an affine map $\bar{g} : |\mathcal{P} \cup \partial P| \rightarrow \mathbf{R}^+ \cup 0$ and use \bar{g} as a height function for $|\mathcal{P} \cup \partial P|$. This gives a two-dimensional polyhedron, denoted by \bar{P}_g , which is homotopically equivalent to \mathcal{P} . We have

$$(4.1) \quad \chi(\bar{P}_g) = \chi(\mathcal{P}) = -1.$$

Double \bar{P}_g along its boundary to obtain a genus 2 closed surface which will be denoted by $D\mathcal{P}$. The Index Theorem (see Theorem 2.3) asserts that

$$(4.2) \quad \sum_{v \in D\mathcal{P}} \text{Ind}_g(v) = -2.$$

It follows from Equation (2.1) and a simple Euler characteristic computation that we must have a single vertex $u \in \bar{P}_g$ whose index equals -1 , which means that $\text{Sgc}(u) = 4$. Let $L(u)$ be the figure eight curve corresponding to the value $\bar{g}(u)$. Then $L(u) = L_I \cup L_{II}$, where L_I and L_{II} are two simple cycles intersecting at u (using here the assertion of Theorem 2.15).

Moreover, since E_2^1 and E_2^2 are the only inner components of $\partial\Omega$, by applying Lemma 2.8 we may assume that E_2^1 is contained in L_I and E_2^2 is contained in L_{II} . By cutting along $L(u)$, we can decompose $\Omega \cup \partial\Omega$ into three components. The first, denoted by Ω_1 , has boundary components E_1 and $L(u)$. The second, denoted by Ω_2^1 , has boundary components E_2^1 and L_I , and the third, denoted by Ω_2^2 , has boundary components E_2^2 and L_{II} . Note that the interiors of these components are all homeomorphic to $S^1 \times (0, 1)$, and moreover that Ω_2^1 and Ω_2^2 are each homeomorphic to an annulus, however $\chi(\Omega_1) = -1$. Therefore, Ω_1 may be viewed as an annulus with one singular boundary component, or equivalently as a really short pair of pants, i.e. one with no cuffs. We now define three D-BVPs on the above domains following the discussion preceding Definition 2.7. Note that an essential property of $L(u)$, which is given by Theorem 2.15, allows us to define a D-BVP on Ω_1 .

Let g_1 be the solution on Ω_1 , g_2^1 on Ω_2^1 and g_2^2 on Ω_2^2 . By Remark 2.41, we have that the lengths of L_I and E_2^1 , with respect to the flux-gradient metric induced by g_2^1 , are both equal to

$$(4.3) \quad \left| \sum_{x \in E_2^1} \frac{\partial g}{\partial n}(\Omega_2^1)(x) \right|.$$

Similarly, we have for Ω_2^2 that the lengths, with respect to the flux-gradient metric induced by g_2^2 of L_{II} and E_2^2 , are equal to

$$(4.4) \quad \left| \sum_{x \in E_2^2} \frac{\partial g}{\partial n}(\Omega_2^2)(x) \right|.$$

Finally, by applying Theorem 2.34 with $C_n = E_1$ and $L = L(u)$, we have that the lengths of E_1 and $L(u)$, with respect to the l_1 gradient metric induced by g_1 , are equal to

$$(4.5) \quad \sum_{x \in E_1} \frac{\partial g}{\partial n}(\Omega_1)(x).$$

We now turn to the construction of $(S_{\mathcal{P}}, f)$. First, apply Theorem 0.4 to Ω_2^1 and to Ω_2^2 . The outputs are two straight Euclidean cylinders $S_{\Omega_2^1}$ and $S_{\Omega_2^2}$, two mappings $f_{\Omega_2^1} : \Omega_2^1 \rightarrow \mathcal{S}_1$ and $f_{\Omega_2^2} : \Omega_2^2 \rightarrow \mathcal{S}_2$ (where the maps, their domains and the cylinders are described in detail

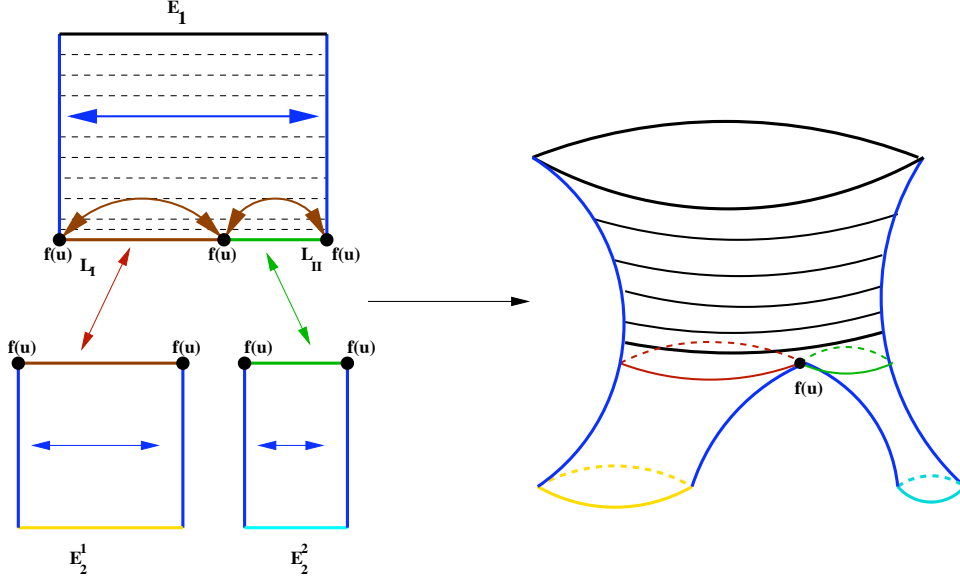


FIGURE 4.6. The construction of the Euclidean pair of pants with one singular point.

in Theorem 0.4). Second, we wish to apply Theorem 0.4 to Ω_1 . However, one boundary component of it ($L(u)$) is singular at u . Note that the construction described in the proof of Theorem 0.4 will go through for any annulus contained in Ω_1 with one boundary component being E_1 and the other being $L_{g(L(u))+\epsilon}$ (with $\epsilon > 0$ arbitrarily small). Thus, we obtain a sequence of Euclidean cylinders that have the same circumference (the flux-gradient metric length of E_1) and their heights converge to $k - g(L(u))$.

As $\epsilon \rightarrow 0$, the level curves corresponding to $g(L(u)) + \epsilon$ converge geometrically to $L(u)$. Hence, the limiting cylinder \mathcal{S}_0 is a Euclidean cylinder with one singular boundary component (bottom). We call one boundary component of this cylinder *top* and the other *bottom*. Specifically, \mathcal{S}_0 is obtained by taking a Euclidean cylinder of height $k - g(L(u))$ and circumference which is equal to the flux-gradient metric length of E_1 , picking two points on one boundary component of it at a distance which equals the flux-gradient metric length of L_I , and identifying them. We will abuse notation and denote this point by u . Since identification occurs only at the boundary, the tiling persists in the limit. We denote by f_{Ω_1} the mapping given by Theorem 0.4, modified on the bottom as above. Let

$$(4.7) \quad f = f_{\Omega_1} \cup f_{\Omega_2^1} \cup f_{\Omega_2^2},$$

where the domain of f is $\Omega_1 \cup \Omega_2^1 \cup \Omega_2^2$ and its image is obtained by gluing isometrically S_1 and S_2 to the bottom in such a way that the intersection consists of only one point (see Figure 4.6), denoted by $f(u)$. By Remark 2.41, we have that the length (measured with the l_1 -induced metric) of the bottom of \mathcal{S}_0 is equal to the sum of the lengths of the tops of the two cylinders \mathcal{S}_1 and \mathcal{S}_2 (one can also obtain this by applying Proposition 1.4 and Equations (4.3)-(4.5) directly). Furthermore, we require the gluing to be consistent; i.e the gluing described above respects rectangles corresponding to edges which cross $L(u)$ (see the Proof of Theorem 0.4). The fact that this can be done may be justified by basically the same

argument appearing before and after Equation (3.5). Note that one consequence of this is that $f(u)$ is uniquely determined (the modified g being harmonic being the key issue).

It is easy to check that the only cone angle is at this point and is equal to 4π . The mappings $f_{\Omega_2^1}, f_{\Omega_2^2}$ are energy preserving in the sense explained in Theorem 0.4; the mapping f_{Ω_1} is also energy preserving (since the identification is done on bottom only). Since the gluing is by isometries, and the cylinders involved meet only at one vertex, the mapping f is energy preserving. □

4.2. The case of an m -connected domain, $m > 3$. In this subsection we provide a proof of Theorem 0.2. The proof is a modification of Theorem 0.1 with some bookkeeping and successive changes of the initial D-BVP.

Proof of Theorem 0.4. Let L_1 be the unique level curve whose existence is guaranteed by Proposition 2.28. Let Ω_1 be the domain bounded by E_1 and L_1 . Observe that Ω_1 is homeomorphic to $S^1 \times (0, 1)$. Suppose that $L_1 = l_1 \cup l_2 \dots \cup l_k$, where each l_i is a simple cycle (for $i = 1, \dots, k$), where any two of these cycles are either disjoint or intersect at a singular vertex in $\mathcal{T}^{(0)}$ (see Theorem 2.15). Let $\mathcal{U} = \{u_1, \dots, u_m\}$ be the set of all singular vertices along L_1 with $\{\text{Ind}_g(u_1), \dots, \text{Ind}_g(u_m)\}$ being their indices, respectively.

The interior of each l_i , which will be denoted by l_i° , is a q_i -connected domain with $q_i < m - 1$, for $i = 1, \dots, k$, (unless $k = 1$). We now modify the initial D-BVP on Ω_1, l_i and l_i° as described in the discussion preceding Definition 2.7, and we obtain $k + 1$ new harmonic functions: g_0 which is defined on E_1, Ω_1 and L_1 , and g_1, \dots, g_k defined on l_1, \dots, l_k and their interiors, respectively.

Recall that modifications are made by adding vertices to $\mathcal{T}^{(0)}$ along the l_i 's and by defining conductance along new edges: those which cross the l_i 's (See Equation (2.6)). By part (5) of Theorem 2.34 and Remark 2.41, we have that the length of L_1 , with respect to the flux-gradient metric induced by g_0 , is equal to the length of E_1 with respect to the flux-gradient metric induced by g_0 . Hence both are equal to the length of E_1 with respect to the initial g metric. We record this as

$$(4.8) \quad \text{length}_{g, l_1}(E_1) = \text{length}_{g, l_1}(L_1).$$

By using g_i in the interior of each l_i , we now choose a singular curve enclosing all of its $q_i - 1$ boundary curves (see Proposition 2.28). The result is a set of singular level curves $\mathcal{W} = \{W_1, \dots, W_n\}$ and the set of singular vertices $\mathcal{V} = \{v_1, \dots, v_p\}$ they contain with their associated indices $\mathcal{I} = \{\text{Ind}_g(v_1), \dots, \text{Ind}_g(v_p)\}$. Each simple cycle of a singular level curve W_j contains $s_j < q_i$ boundary curves unless W_j° is an annulus. As above, we have only added vertices on the l_i 's, assigning each one of the vertices on a specific l_i , the $g(l_i)$ -value. The conductance constants are changed only along new edges, those which cross l_i , according to Equation (2.6). Hence, the set \mathcal{W} coincides with the set of singular level curves of g minus L_1 (similar statements hold for \mathcal{V} and \mathcal{I}).

We repeat this process (at most) finitely many times, modifying (if needed) the cellular decomposition and defining conductance constants according to Equation (2.6), until the interior of each simple cycle of each singular level curve is an annulus. At each step, we obtain new harmonic functions defined on domains with fewer boundary components. An equation analogous to Equation (4.8) holds for each simple cycle which is a component of a singular level curve and the nearest singular level curve it contains. That is, its length

measured by the flux-gradient metric induced by the harmonic functions defined on its interior is equal to the length of the singular level curve, both measured by the harmonic map defined in the interior of the simple cycle (see part (5) of Theorem 2.34). In turn, they are equal to the length of the simple cycle measured by the harmonic function defined on its exterior (see Theorem 2.20).

It is evident from the proof of Proposition 2.28, the disjointness of level curves which correspond to different g -values, and the maximum principle that there is a well-defined hierarchy on the set of singular level curves. Each component of a singular level curve, say $g^{-1}(k_m)$, $k_m \in \mathcal{K}$, is contained in the interior of a domain whose one component is a unique simple cycle. This cycle belongs to a unique singular curve, say $g^{-1}(k_n)$, $k_n \in \mathcal{K}$, $k_n > k_m$ (or E_1). Simple cycles corresponding to level curves are either contained in a domain bounded by a simple cycle which is part of a singular level curve, or are parts of singular level curves as is detailed in Theorem 2.15.

We now turn to the construction of S_Ω and the mapping f . We start with a straight Euclidean cylinder, C_{E_1, L_1} , of height $k - g(L_1)$ and circumference which is equal to

$$(4.9) \quad \sum_{x \in E_1} \frac{\partial g}{\partial n}(x).$$

Since L_1 is a generalized bouquet of circles, we can select a finite number of points in the bottom and identify subsets of them in such a way that the quotient is topologically and metrically isomorphic to L_1 .

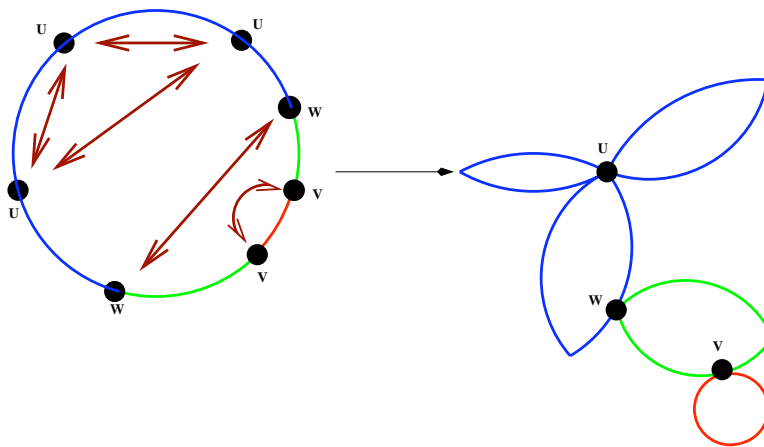


FIGURE 4.10. A quotient of a boundary component of a Euclidean cylinder yielding a generalized bouquet of circles

As in the proof of Theorem 0.1, we wish to apply Theorem 0.4 to $\Omega_1 \cup \partial\Omega_1$ ($\partial\Omega_1 = E_1 \cup L_1$). Since L_1 is singular, we may not apply it directly. Instead, we follow the argument carried out in the proof of Theorem 0.1. All level curves (which are simple) corresponding to $g(L_1) + \epsilon$, with $\epsilon > 0$ being small, will have the same length as L_1 (see part (5) of Theorem 2.34); hence, as $\epsilon \rightarrow 0$, the sequence of Euclidean cylinders (guaranteed by Theorem 0.4 applied to the sequence of annuli bounded by these level curves and E_1) with their tops being fixed, heights equal to $k - (g(L_1) + \epsilon)$, and circumferences equal to $\sum_{x \in E_1} \frac{\partial g}{\partial n}(x)$, converges to \tilde{C}_{E_1, L_1} with

the top persisting in the limit and the bottom replaced by its quotient as explained above. We denote the mapping given by Theorem 0.4 and modified as above on the bottom by f_{Ω_1} .

For each l_i , $i = 1, \dots, k$, there are two cases to consider. First, suppose that l_i° does not contain any singular level curve. Without loss of generality, let the unique boundary curve it contains be E_2^i . By assertion (4) of Theorem 2.34, we conclude that the lengths of E_2^i and l_i are equal (measured with the g_i -induced flux-gradient metric). Thus, $\mathcal{A}_i = l_i^\circ \cup l_i \cup E_2^i$ is an annulus with two (non-singular) boundary components. We may therefore apply Theorem 0.4 and obtain a mapping $f_i : \mathcal{A}_i \rightarrow C_{l_i, E_2^i} (= S_{\mathcal{A}_i})$. We now attach the top of the resulting Euclidean cylinder C_{l_i, E_2^i} to l_i by an isometry which is consistent. We conclude that the gluing may be taken as such, by first arguing that the length of l_i measured with respect to the flux-gradient metric induced by g_0 , is the same as its length with respect to the length induced by the flux-gradient metric of g_i . This is justified directly by employing Theorem 2.20 or by using a series of equalities similar to those in Equations (4.3)-(4.5) and the reasoning leading to them. The fact that the gluing is consistent follows from the same arguments we have used before.

Second, assume that l_i° contains a singular curve, say $W_i \in \mathcal{W}$. Recall that W_i is the unique singular curve in l_i° which encloses all of the boundary curves in l_i° . With l_i replacing E_1 , W_i replacing L_1 , and g_i replacing g_0 , we repeat the construction, yielding a Euclidean cylinder with its bottom being a singular curve as described above (in the case of L_1), and a mapping which we will denote by f_{l_i, W_i} . The cylinder \tilde{C}_{l_i, W_i} is attached to \tilde{C}_{E_1, L_1} by gluing its top to the simple cycle corresponding to l_i in the quotient of the bottom of \tilde{C}_{E_1, L_1} , that is, the singular component of \tilde{C}_{E_1, L_1} .

The above analysis is now carried out, at most finitely many times, for each l_i , $i = 1, \dots, k$, until we are left with Euclidean cylinders having both of their boundary components non-singular. In particular, the bottom of each cylinder is the image (under the map given by Theorem 0.4) of E_2^i and its length is equal to

$$(4.11) \quad - \sum_{x \in E_2^i} \frac{\partial g}{\partial n}(x), \text{ for } i = 1, \dots, m-1.$$

By construction, the images of E_1 and E_2^i , $i = 1, \dots, m$, under the maps whose construction is described above, are the only boundary components of S_Ω . The map f is the obvious union of the collection of mappings constructed above and is analogous to the one appearing in Equation (4.7). It is also evident that f is energy preserving in the sense described before.

We now compute the cone angle, $\phi(v)$, at a singular vertex v with $\text{Ind}_g(v) = -n$, $n \in \mathbf{N}$. By construction, v is the (unique) tangency point of $n+1$ Euclidean cylinders, where v belongs to the non-singular component of each such cylinder. Hence, such a cylinder contributes π to the cone angle at v . Recall that each such cylinder is attached (by an isometry) to the quotient of the bottom of another Euclidean cylinder. It is easy to check that the contribution to the cone angle at v from this Euclidean cylinder (with one singular boundary component) is equal to $n \times \pi + \pi$. Therefore,

$$(4.12) \quad \phi(v) = 2(n+1)\pi.$$

□

Remark 4.13. One may also prove Theorem 0.2 by an induction on the number of boundary components. However, the assertion of Proposition 2.28 must be used as well as an extension of g_0 to g_1 over the singular curve L_1 . Theorem 2.20 needs to be used in order to prove equality of the l_1 -length of L_1 according to both the g_0 and the g_1 metric. Overall, we found the proof which does not use induction conceptually more gratifying. We could also stop the process once a planar pair of pants is encountered, thus using Theorem 0.1 directly. Finally, Theorem 0.1 is of course a special case of the theorem above. Still, we maintain that this special case deserves its own proof.

Remark 4.14. There is a technical difficulty in our construction if some pair of adjacent vertices of $\mathcal{T}^{(0)}$ has the same g -value (the first occurrence is in Equation (2.1)). One may generalize the definitions and the index formula to allow rectangles of area zero, as one solution. For a discussion of this approach and others see [23, Section 5]. Experimental evidence shows that when the cell decomposition is complicated enough, even when the conductance function is identically equal to 1 and the cells are triangles, such equality rarely happens.

Remark 4.15. The existence of singular curves for g results in the fact that some rectangles are not embedded in the target. This is evident by the proofs of Theorem 0.1 and Theorem 0.2. Since some of the cylinders constructed have a singular boundary component, it is clear that some points in different rectangles that lie on this level curve will map to the same point. However, this occurs only in the situation above and since this fact is not of essential interest to us, we will not go into more details.

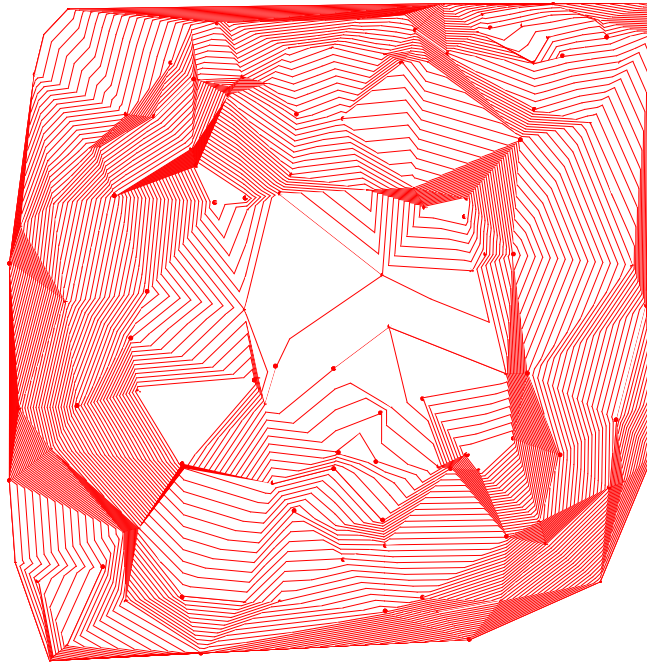


FIGURE 4.16. Level curves for a 5 connected domain

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