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Energy and length in a topological planar quadrilateral

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Abstract

We provide bounds for the product of the lengths of distinguished shortest paths in a finite network induced by a triangulation of a topological planar quadrilateral.

1. Introduction

A topological planar closed disk with four distinguished points on its boundary, its corners, will be called a quadrilateral. The following definition is due to Schramm [16].

Definition 1.1. Let Q be a quadrilateral endowed with a triangulation. Let V, E, T denote the set of vertices, edges and triangles of Q, respectively. Let $\partial Q = P_1 \bigcup P_2 \bigcup P_3 \bigcup P_4$ be a decomposition of ∂Q into four non-trivial arcs of the triangulation with disjoint interiors, in cyclic order. If the intersection of two any of these arcs is not empty, then it consists of a corner (all of the corners are vertices). A corner must belong to one and only one of the P_i's. The collection $\mathcal{T} = (V, E, T, P_1, P_2, P_3, P_4)$ will be called a triangulation of Q.

On invoking a *conductance function*, $\mathcal{T}$ becomes a *finite network*. One can define a *boundary value problem* (BVP) on the network. Let f be the solution of the BVP and let $I(f)$ be its *Dirichlet energy*. Corollary 3.7 provides inequalities relating the product of the lengths of a shortest *thick* vertical path (a particular path which connects P_3 and P_1) and a shortest thick horizontal path (a particular path which connects P_2 and P_4) in terms of $I(f)$ and some constants arising from the combinatorics and the conductance function. Corollary 3.7 follows from Theorem 3.5 (our main theorem) and Lemma 3.6. The length is measured with respect to ρ,

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the *gradient metric* (Definition 3.3) which is induced by the solution of the BVP (see Sections 2 and 3 for the precise definitions of the notions above). In the special case where $c(x, y) \equiv 1$ and k is the maximal degree of V, it follows from Corollary 3.7 that

$$l(|V|)I(f) \geq \text{length}_\rho(\gamma^*)\text{length}_\rho(\gamma) \geq \frac{1}{\sqrt{k}} I(f), \quad \text{where } l(|V|) \text{ is some constant.}$$

Loewner (see [7]) studied differential–geometric inequalities relating area and the product of shortest (vertical and horizontal) curves in a quadrilateral. His inequalities are derived with respect to the Euclidean metric. His work was generalized and forms a rich theory. The well known reciprocal property of the *extremal lengths* of conjugate families of curves in a quadrilateral is one useful example of this theory (see [1,14,15] for a few examples and generalizations for other Riemann surfaces).

The original notion of extremal length in a discrete setting was introduced by Duffin [11]. More recently, in this setting, Cannon [8] introduced a different notion of extremal length. In the work of Cannon, Floyd and Parry (see for instance [9]), as well as in the work of Schramm [16], inequalities generalizing the reciprocal property of the extremal length (with respect to some extremal metric) of conjugate families of curves in a quadrilateral are very useful.

In the setting of finite and infinite networks the reciprocal properties of extremal length and *capacity* were studied extensively with respect to an extremal metric (see [18] for a detailed account).

One motivation for using the gradient metric in this paper arises from extremal length arguments in the complex plane. It is well known (see [1]) that for every $z \in \mathbf{C}$ the extremal metric in a topological quadrilateral in the complex plane satisfies

$$m(z) = |\nabla(f)(z)|,$$

where f is the solution of the classical Dirichlet–Neumann boundary value problem. In the complex plane it is also known that equality holds in Corollary 3.7, where both sides equal $I(f)$.

Consider the broader class of BVP problems that are studied in [2]. Our work is also motivated by the following.

Question. Is there a BVP problem and a metric ρ_0 (which is perhaps different than ρ) derived from the solution such that

$$\text{length}_{\rho_0}(\gamma^*)\text{length}_{\rho_0}(\gamma) \geq I(f) \quad \text{and} \quad \sum_{x \in \bar{F}} \rho_0^2(x) = I(f) \text{ (see Definition 3.2)?}$$

Remark. In the paper [13] we use some of the ideas of this paper to prove a finite Riemann mapping theorem [9,16]. A more direct proof would follow from a positive answer to the question above. A finite Riemann mapping theorem can be viewed as the first step in solving the Cannon conjecture: A negatively curved group G with $\partial G = S^2$ is Kleinian. We hope that our ideas will be useful towards the resolution of this conjecture.

2. Preliminaries

We recall some known facts regarding harmonic functions and boundary value problems on networks. We use the notation of Section 2 in [2]. Let $\Gamma = (V, E, c)$ be a *finite network*, that is a simple and finite connected graph with a vertex set V and edge set E. We shall also assume that the graph is planar. Each edge $(x, y) \in E$ is assigned a *conductance* $c(x, y) = c(y, x) > 0$. Let $\mathcal{P}(V)$ denote the set of non-negative functions on V. If $u \in \mathcal{P}(V)$, its support is given by $S(u) = \{x \in V : u(x) \neq 0\}$. Given $F \subset V$ we denote by F^c its complement in V. Set $\mathcal{P}(F) = \{u \in \mathcal{P}(V) : S(u) \subset F\}$. The set $\partial F = \{(x, y) \in E : x \in F, y \in F^c\}$ is called the *edge boundary* of F and the set $\delta F = \{x \in F^c : (x, y) \in E$ for some $y \in F\}$ is called the *vertex boundary* of F. Let $\bar{F} = F \bigcup \delta F$ and let $\bar{E} = \{(x, y) \in E : x \in F\}$. Given $F \subset V$, let $\bar{\Gamma}(F) = (\bar{F}, \bar{E}, \bar{c})$ be the network such that $\bar{c}$ is the restriction of c to $\bar{E}$. We say that $x \sim y$ if $(x, y) \in \bar{E}$. For $x \in \bar{F}$ let $k(x)$ denote the degree of x (if $x \in \delta(F)$ the neighbors of x are taken only from F).

For $f, h : \bar{E} \to R$ we let $(f, h) = \sum_{(x,y)\in\bar{E}} \frac{f(x,y)h(x,y)}{c(x,y)}$ be an inner product on $l^2(\bar{E}, 1/c)$ (see [19, 1.2.A]). The following definitions are discrete analogues of classical notions in continuous Potential Theory [12].

Definition 2.1 (*[3, Section 3]*). Let $u \in \mathcal{P}(\bar{F})$;

1. then for $x \in \bar{F}$, the function $\Delta u(x) = \sum_{y\sim x} c(x, y)(u(x) - u(y))$ is called the potential of u at x (if $x \in \delta(F)$ the neighbors of x are taken only from F), and
2. the number $I(u) = \sum_{x\in\bar{F}} \Delta u(x)u(x) = \sum_{(x,y)\in\bar{E}} c(x, y)(u(x) - u(y))^2$ is called the *Dirichlet energy* of u;
3. a function $u \in \mathcal{P}(\bar{F})$ is called harmonic in $F \subset V$ if $\Delta u(x) = 0$, for all $x \in F$.

When $c(x, y) \equiv 1$, an easy computation shows that u is harmonic at a vertex x if and only if the value of u at x is the arithmetic average of the value of u on the neighbors of x.

When $(x, y) \in \bar{E}$ let us denote by $[x, y]$ the directed edge from x to y and let $\overrightarrow{E} = \{[x, y] : (x, y) \in \bar{E}\}$ denote the set of all directed edges. Given $u : V \to \mathbf{R}$ we define the *differential* or the *gradient* of u as $\mathrm{d}u : \overrightarrow{E} \to \mathbf{R}$ by $\mathrm{d}u[x, y] = c(x, y)(u(y) - u(x))$ for all $[x, y] \in \overrightarrow{E}$ (see for instance the notation of Section 2 in [6]). Note that if $|\overrightarrow{E}| = m$, then $\mathrm{d}u$ can be identified with a vector in $\mathbf{R}^m$. It now follows by Definition 2.1 that for every function $u : V \to \mathbf{R}$ we have that $I(u) = \frac{1}{2} \sum_{e\in\overrightarrow{E}} \|\mathrm{d}u(e)\|^2$.

Let $\overrightarrow{E}(x)$ denote the set of all edges of the form $[x, y]$ which are in $\overrightarrow{E}$. Any $g : \overrightarrow{E}(x) \to \mathbf{R}$ can be naturally viewed as an element in $\mathbf{R}^{k(x)}$. We will denote this vector space, with the restriction of the inner product on $\bar{E}$, by T_x. In particular we have

Definition 2.2. Let $u : \bar{F} \to \mathbf{R}$. Let $\overrightarrow{\mathrm{d}u}(x) = \mathrm{d}u|_{\overrightarrow{E}(x)}$ denote the restriction of $\mathrm{d}u$ to $\overrightarrow{E}(x)$ (in particular, $\overrightarrow{\mathrm{d}u}(x)$ can be viewed as a vector in T_x).

For $x \in \delta(F)$ let $\{y_1, y_2, \ldots, y_m\} \in F$ be its neighbors, enumerated in a cyclic order.

Definition 2.3. The normal vector derivative at $x \in \delta(F)$ is defined by $\overrightarrow{\frac{\partial u}{\partial n_F}}(x) = (c(x, y_1)(u(x) - u(y_1)), \ldots, c(x, y_m)(u(x) - u(y_m)))$ and the conductance vector at x is defined by $\overrightarrow{c}_{\delta(F)}(x) = (c(x, y_1), \ldots, c(x, y_m))$.

If $x \in F$, $\overrightarrow{c}_F(x)$ is defined similarly and the neighbors of x are taken in $F \bigcup \delta(F)$.

The following definition provides the discrete analogue of the continuous notion of *normal derivative*.

Definition 2.4 (*[10]*). The normal derivative of u at a point $x \in \delta F$ with respect to the set F is

$$\frac{\partial u}{\partial n_F}(x) = \sum_{y\sim x, y\in F} c(x, y)(u(x) - u(y)).$$

The following proposition establishes a discrete version of the first classical *Green identity*. It will be crucial in the proof of Theorem 3.5.

Proposition 2.5 (*[2, Proposition 3.1] (The First Green Identity)*). *Let $F \subset V$ and $u, v \in \mathcal{P}(\bar{F})$. Then we have that*

$$\sum_{(x,y)\in\bar{E}} c(x,y)(u(x)-u(y))(v(x)-v(y)) = \sum_{x\in F} \Delta u(x)v(x) + \sum_{x\in\delta(F)} \frac{\partial u}{\partial n_F}(x)v(x).$$

Remarks. 1. In [2] a second Green identity is obtained. In this paper we will use only the one above.

2. In [5] (see in particular Sections 2 and 3) a systematic study of discrete calculus on n-dimensional (uniform) grids of Euclidean n-space is provided. Their definition of a tangent space may be adapted to our setting and does not require the notion of directed edges. However, in [13] directed edges will play an important role.

3. Length estimates of shortest paths

Throughout this section $\mathcal{T}$ will denote a fixed triangulation of a quadrilateral (see Definition 1.1). We will denote by F the set of vertices which do not belong to ∂Q. Hence, $\delta(F)$ is the set of vertices that belong to $P_1 \bigcup P_2 \bigcup P_3 \bigcup P_4$. Let $\{c(x,y)\}_{(x,y)\in\bar{E}}$ be a fixed conductance function and let $\bar{\Gamma}(F)$ be the associated network. We are interested in functions that solve a boundary value problem (BVP) on $\bar{\Gamma}(F)$. The following definition is based on [2, Section 3] and [4, Section 4].

Definition 3.1. Let $g > 0$ be a constant. A Dirichlet–Neumann boundary value function is a function $f \in \mathcal{P}(\bar{F})$ which satisfies the following:

1. f is harmonic in F,
2. $f|_{P_2} = 0$,
3. $f|_{P_4} = g$, for some constant g, and
4. $\frac{\partial f}{\partial n_F}|_{P_1} = \frac{\partial f}{\partial n_F}|_{P_3} = 0$.

Remark. The uniqueness and existence of a Dirichlet–Neumann boundary value function is provided by the nice and foundational work in [2, Section 3] and [4, Section 4]. In fact, their work provides a detailed framework for a broader class of boundary value problems on finite networks.

Definition 3.2 (*[8]*). A metric on a finite network is a function $\rho : V \to [0, \infty)$.

In particular, the length of a path is given by integrating ρ along the path. When $\rho \equiv 1$, the familiar distance function on $V \times V$ is obtained by setting $\text{dist}(A, B) = \sum_{x\in\alpha} 1 - 1 = k$, where $\alpha = (x, x_1, \dots x_k)$ is a path with the smallest possible number of vertices among all the paths connecting a vertex in A and a vertex in B. We now define the gradient metric which will be used in our estimates.

Definition 3.3. Given $f \in \mathcal{P}(\bar{F})$ the gradient metric induced by $f \in \mathcal{P}(\bar{F})$ is defined by

$$\rho(x) = \begin{cases} \|\overrightarrow{\mathrm{d}f}(x)\| & \text{if } x \in F \\ \left\|\dfrac{\overrightarrow{\partial f}}{\partial n_F}(x)\right\| & \text{if } x \in \delta(F). \end{cases}$$

Before turning to our main theorem, we will define the paths which are going to be considered. The definition below describes two classes of paths. These classes are sufficiently separated from $\delta(F)$.

Definition 3.4. A path $\beta = (x_0, x_1, \ldots, x_n)$ will be called vertically (horizontally) thick if it satisfies the following:

1. $x_0 = \beta \cap P_3$ ($x_0 = \beta \cap P_2$) and $x_n = \beta \cap P_1$ ($x_n = \beta \cap P_4$) respectively,
2. neither x_0 or x_n is a corner,
3. for all $i = 1, \ldots n-1$, $x_i \in F$,
4. for all $i = 2, \ldots n-2$, $\mathrm{dist}(x_i, P_3 \cup P_4 \cup P_1) > 1$,
5. $\mathrm{dist}(x_1, x), \mathrm{dist}(x_{n-1}, x) \geq 1$ when $x \in P_1 \cup P_3 \cup P_4$ ($\mathrm{dist}(x_1, x), \mathrm{dist}(x_{n-1}, x) \geq 1$ when $x \in \delta(F)$), and equality is attained uniquely for x_0 and x_n respectively.

We now turn to our main theorem.

Theorem 3.5. *Let $\mathcal{T}$ be a triangulation of a topological quadrilateral. Let $\bar{\Gamma}(F)$ be the associated network. Let f be the Dirichlet–Neumann boundary value function with some constant g. Let ρ be the gradient metric induced by f. Let* $M = \max_{x \in F \cup \delta(F)}\{\|\overrightarrow{c}_F(x)\|, \|\overrightarrow{c}_{\delta(F)}(x)\|\}$ *and let* $m = \min_{(x,y)\in\bar{E}} \sqrt{c(x,y)}$.

1. *If γ^* is a shortest vertical thick path which connects P_1 to P_3, then*
$$\mathrm{length}_\rho(\gamma^*) \geq \frac{I(f)}{gM}, \textit{ and}$$
2. *if γ is a shortest horizontal thick path which connects P_2 to P_4, then*
$$\mathrm{length}_\rho(\gamma) \geq gm.$$

Proof. Using properties (1)–(4) of f and the first Green identity (Proposition 2.5) with $u = v = f$ we obtain that

$$I(f) = \sum_{x \in P_4} \frac{\partial f}{\partial n_F}(x) f(x). \tag{1}$$

Hence, by the definition of g, we have that

$$I(f) = g\left|\sum_{x \in P_4} \frac{\partial f}{\partial n_F}(x)\right|. \tag{2}$$

Let γ^* be a shortest thick path connecting P_3 to P_1. It is clear that we may assume that γ^* is simple. Let $x_0 = \gamma^* \cap P_3$ and let $x_n = \gamma^* \cap P_1$. (See Fig. 1.)

Let V_{γ^*} denote the subset of F which is enclosed, in cyclic order, by γ^*, a part of P_1 (which we will denote by $P_1(\gamma^*)$), P_4 and part of P_3 (which we will denote by $P_3(\gamma^*)$). It follows from Definition 3.4 that $V_{\gamma^*} \neq \emptyset$ and that $\delta(V_{\gamma^*}) = \gamma^* \cup P_1(\gamma^*) \cup P_4 \cup P_3(\gamma^*)$.

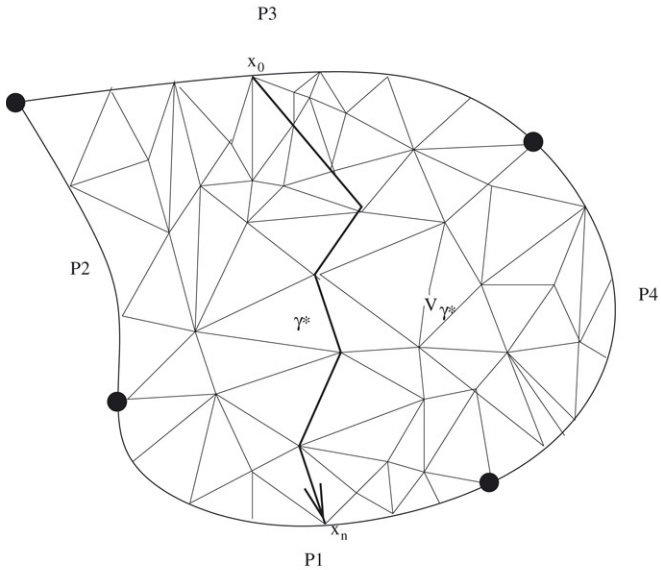

Fig. 1. A shortest thick vertical path between P_3 and P_1.

We now apply the first Green identity with $u = f$ and the constant function $v \equiv 1$ in $V_{\gamma^*} \cup \delta(V_{\gamma^*})$. We obtain that

$$\sum_{x \in \delta(V_{\gamma^*})} \frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) = 0. \tag{3}$$

It follows by Definition 3.4 that for every $x \in P_1(\gamma^*) \cup P_4 \cup P_3(\gamma^*)$ which is not in γ^* we have that

$$\frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) = \frac{\partial f}{\partial n_F}(x).$$

By using the fourth property of Definition 3.1 and the triangle inequality we have that

$$\left| \sum_{x \in P_4} \frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) \right| = \left| \sum_{x \in \gamma^*} \frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) \right| \leq \sum_{x \in \gamma^*} \left| \frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) \right|. \tag{4}$$

For every $x \in \gamma^*$ (viewed now as a vertex in $\delta(V_{\gamma^*})$) we have that

$$\frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) = \left(\vec{c}_{\delta(V_{\gamma^*})}(x), \overrightarrow{\frac{\partial f}{\partial n_{V_{\gamma^*}}}}(x) \right).$$

Hence by the Cauchy–Schwartz inequality we have that

$$\left| \frac{\partial f}{\partial n_{V_{\gamma^*}}}(x) \right| = \left| \left(\vec{c}_{\delta(V_{\gamma^*})}(x), \overrightarrow{\frac{\partial f}{\partial n_{V_{\gamma^*}}}}(x) \right) \right| \leq \| \vec{c}_{\delta(V_{\gamma^*})}(x) \| \left\| \overrightarrow{\frac{\partial f}{\partial n_{V_{\gamma^*}}}}(x) \right\|. \tag{5}$$

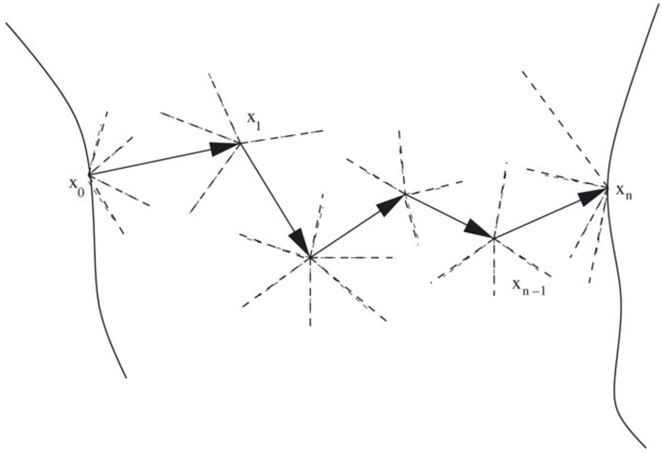

Fig. 2. A shortest thick horizontal path between P_2 and P_4.

It is also clear that for every $x \in \gamma^*$ which is different from x_0 or x_n, we have that

$$\|\vec{c}_{\delta(V_{\gamma^*})}(x)\| \le \|\vec{c}_F(x)\| \quad \text{and} \quad \left\| \overrightarrow{\frac{\partial f}{\partial n_{V_{\gamma^*}}}}(x) \right\| \le \rho(x). \tag{6}$$

If $x = x_0$ or $x = x_n$ we have that

$$\|\vec{c}_{\delta(V_{\gamma^*})}(x)\| \le \|\vec{c}_{\delta(F)}(x)\| \quad \text{and} \quad \left\| \overrightarrow{\frac{\partial f}{\partial n_{V_{\gamma^*}}}}(x) \right\| \le \rho(x). \tag{7}$$

Hence, by summing over all $x \in \gamma^*$, the definition of M, Eqs. (2) and (4)–(7), we have that

$$\frac{I(f)}{gM} \le \sum_{x \in \gamma^*} \rho(x) = \text{length}_\rho(\gamma^*), \tag{8}$$

which is first assertion of the theorem.

Let γ be a shortest thick path connecting P_2 and P_4. It is clear that we may assume that γ is simple. Let $x_0 = \gamma \cap P_2$ and let $x_n = \gamma \cap P_4$. (See Fig. 2).

By integrating ρ along γ we have that

$$\sum_{x \in \gamma} \rho(x) = \left\| \overrightarrow{\frac{\partial f}{\partial n_F}}(x_0) \right\| + \|\mathrm{d}f(x_1)\| + \|\mathrm{d}f(x_2)\| + \cdots + \|\mathrm{d}f(x_{n-1})\| + \left\| \overrightarrow{\frac{\partial f}{\partial n_F}}(x_n) \right\|.$$

For each $i = 1, \ldots, n-1$ we have that

$$\|\mathrm{d}f(x_i)\| \ge \sqrt{c(x_i, x_{i+1})}|f(x_{i+1}) - f(x_i)|.$$

It is easy to see that $\left\| \overrightarrow{\frac{\partial f}{\partial n_F}}(x_0) \right\| \ge \sqrt{c(x_0, x_1)}|f(x_1) - f(x_0)|$ and that $\left\| \overrightarrow{\frac{\partial f}{\partial n_F}}(x_n) \right\| \ge \sqrt{c(x_{n-1}, x_n)}|f(x_n) - f(x_{n-1})|$. We now sum over all x_i, use the definition of m and use the triangle inequality to obtain that

$$\sum_{x \in \gamma} \rho(x) \ge \sum_{i=0}^{n-1} \sqrt{c(x_i, x_{i+1})}|f(x_{i+1}) - f(x_i)|$$

$$\geq m|f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})| \geq mg. \tag{9}$$

Assertion (2) of the theorem now follows. □

Remark. It is easy to check that Assertion (2) of the theorem will hold for a larger class of horizontal paths.

We now provide an upper bound for the product of the lengths of any shortest paths in the network.

Lemma 3.6. *Let $\mathcal{T}$ be a triangulation of a topological quadrilateral. Let $\bar{\Gamma}(F)$ be the associated network. Let f be the Dirichlet–Neumann boundary value function with some constant g. Let ρ be the gradient metric induced by f. Then for any ρ shortest curves α, β in $\bar{\Gamma}(F)$ we have that*

$$\text{length}_\rho(\alpha)\text{length}_\rho(\beta) \leq l(|V|)I(f),$$

where $l(|V|)$ is some constant which depends on $|V|$.

Proof. Let $\alpha = (x_0, x_1, \ldots, x_n)$ be a ρ shortest curve in $\bar{\Gamma}(F)$ connecting the vertex x_0 to the vertex x_n. Then $\text{length}_\rho(\alpha) = \sum_{x\in\alpha} \rho(x)$. By the definition of ρ (Definition 3.3) we have for all $x \in \bar{F}$ that

$$\rho(x) = \left(\sum_{y\in\bar{F}} c(x, y)(f(x) - f(y))^2\right)^{1/2} \leq \left(\sum_{(x,y)\in\bar{E}} c(x, y)(f(x) - f(y))^2\right)^{1/2} = \sqrt{I(f)}.$$

It follows from Chapter 31 in [17] (with only minor changes needed in our setting) that $n = O((|E| + |V|)\log|V|)$. Since $\bar{\Gamma}(F)$ is planar we also have that $|E| = O(|V|)$. Hence we have that $n = O(|V|\log|V|)$. Therefore it follows that

$$\text{length}_\rho(\alpha) = O(|V|\log|V|)\sqrt{I(f)}.$$

The assertion of the lemma follows easily. □

Corollary 3.7. *Under the assumptions of Theorem 3.5 and Lemma 3.6 we have that*

$$l(|V|)I(f) \geq \text{length}_\rho(\gamma^*)\text{length}_\rho(\gamma) \geq \frac{m}{M} I(f).$$

Remark. In the case where $c(x, y) \equiv 1$, it is easy to see that $\frac{m}{M} = \frac{1}{\sqrt{k}}$.

4. An example

With the triangulation as given in Fig. 3, let us solve the Dirichlet–Neumann boundary value (Definition 3.1) with $c(x, y) \equiv 1$ and $g = 1$. By abuse of notation let us use the same letter to indicate both the vertex name and the value of the solution at this vertex. Simple calculations (performed with Mathematica) show the following.

1. The solution is $(X, V, S, T, Y, L, U, C1, C2, C3, C4) = (\frac{1}{2}, \frac{1}{2}, \frac{31}{44}, \frac{13}{44}, \frac{13}{44}, \frac{31}{44}, \frac{1}{2}, \frac{3}{11}, \frac{1}{2}, \frac{8}{11}, \frac{1}{2})$,

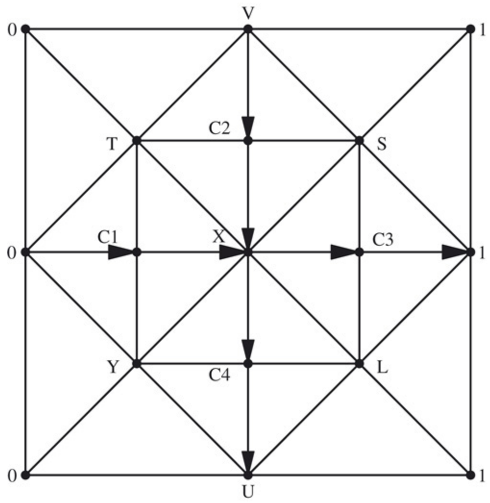

Fig. 3. An example.

2. $[0, C1, X, C3, 1]$ is the shortest thick horizontal geodesic and its length, denoted by l_h, with respect to the gradient metric is approximately 2.23111,
3. $[V, C2, X, C4, U]$ is the shortest thick vertical geodesic and its length, denoted by l_v, with respect to the gradient metric is approximately 1.67733,
4. the energy, $I(f)$ of the solution equals $16/11 \sim 1.45455$,
5. $k = 8$, and
6. $\frac{l_h l_v}{I(f)} - \frac{1}{\sqrt{8}} \sim 2.21929 > 0$.

We conclude that the lower bound provided by the remark following Corollary 3.7 is not sharp.

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