Ergod. Th. & Dynam. Sys. (2004), **24**, 803–824
DOI: 10.1017/S0143385703000300

Counting orbit points in coverings of negatively curved manifolds and Hausdorff dimension of cusp excursions

SA'AR HERSONSKY† and FRÉDÉRIC PAULIN‡

† *Department of Mathematics, Ben Gurion University, Beer-Sheva, Israel*
(e-mail: saarh@math.bgu.ac.il)
‡ *Département de Mathématique et Applications, UMR 8553 CNRS,*
École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France
(e-mail: Frederic.Paulin@ens.fr)

(*Received* 7 *January* 2003 *and accepted in revised form* 30 *July* 2003)

Abstract. We study the growth of fibers of coverings of pinched negatively curved Riemannian manifolds. The applications include counting estimates for horoballs in the universal cover of geometrically finite manifolds with cusps. Continuing our work on diophantine approximation in negatively curved manifolds started in an earlier paper (*Math. Zeit.* **241** (2002), 181–226), we prove a Khintchine–Sullivan-type theorem giving the Hausdorff measure of the geodesic lines starting from a cusp that are well approximated by the cusp returning ones.

1. *Introduction*

Let M be a complete pinched negatively curved Riemannian manifold. Let $\pi : N \to M$ be a Riemannian covering of M and let x_0 be any point in N. Define the *counting function of* π as $f_\pi : \mathbb{R}_+ \to \mathbb{N}$, with $f_\pi(t)$ the number of points x in $\pi^{-1}\pi(x_0)$ such that $d_N(x_0, x) \leq t$. In this paper, we study the growth of the counting function f_π. When π is the covering defined by a cuspidal subgroup of $\pi_1 M$, we get estimates for the growth of the number of horoballs in the universal cover of geometrically finite manifolds with cusps. An application is given to a Khintchine–Sullivan-type theorem in the setting of diophantine approximation in negatively curved manifolds as developed in [**HP2**].

Let $\widetilde{\pi} : \widetilde{M} \to M$ be a universal covering. The growth of $f_{\widetilde{\pi}}$ has been much studied. In particular, the estimate $f_{\widetilde{\pi}}(t) \sim ce^{\delta t}$ as t tends to $+\infty$ has been established in several situations, where δ is the critical exponent of M (see §2) and $c > 0$ is some constant. See [**Mar**] for M compact; see [**Pat, Sul1**] for M non-elementary, geometrically finite, with constant curvature; see [**Rob**] for M non-elementary, having finite Bowen–Margulis measure and with the length spectrum of M being non-discrete in $\mathbb{R}$; see other references in [**Rob**]. When M is compact and π is normal, then the growth of f_π is equivalent to the growth of the finitely generated group $\pi_1 M/\pi_*(\pi_1 N)$ (see [**GK**]). We are mostly interested in non-normal covers.

Recall that N is *convex-cocompact* if N contains a compact convex submanifold which is a strong deformation retract of N. Given two maps $f, g : E \to \mathbb{R}$, write $f \asymp g$ if there is a constant $c > 0$ such that $(1/c)f(t) \le g(t) \le cf(t)$ for every t in E.

THEOREM 1.1. *Assume that M is compact, N is convex-cocompact and π is infinite-sheeted. Then $f_\pi(t) \asymp e^{ht}$, where h is the topological entropy of the geodesic flow of M.*

See Theorem 3.1 for a more general result (valid for Gromov-hyperbolic metric spaces).

Assume for simplicity in this introduction that M is geometrically finite and has exactly one cusp e (see [**Bow**] or §2 for definitions).

As defined in [**HP2**, Definition 2.3], a geodesic line starting from e is *rational* if it converges to e and *irrational* if it accumulates inside M. The *depth* $D(r)$ of a rational line r is the length of the subsegment of r between the first and last meeting point with the boundary of the maximal Margulis neighborhood of the cusp e in M. For $t \ge 0$, define $\mathcal{N}_e(t)$ as the number of rational lines whose depth is less than t.

Let $M_0 \to M$ be the covering of M defined by any parabolic subgroup corresponding to e of the fundamental group of M, let $\widetilde{\pi}_0 : \widetilde{M} \to M_0$ be its universal cover and let δ_0 be its critical exponent. In variable curvature, the geometry of M_0 can be quite complicated (see [**DOP**]). The assumptions $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$ and $\delta_0 < \delta$ are satisfied, for instance, if M is a rank 1 locally symmetric space. See [**DOP**] for many other cases, as well as for situations when they do not hold.

We improve the main result of [**BHP**], as follows (see also [**Rob**]).

THEOREM 1.2. *If $\delta_0 < \delta$, then $\mathcal{N}_e(t) \asymp e^{\delta t}$.*

The main goal of this paper is the following result. Let ξ_0 be any parabolic fixed point corresponding to e. If r is a rational line, let $\widetilde{r}(\infty)$ be the point at infinity of a lift of r starting from ξ_0. Let $\widetilde{d}_e$ be the Hamenstädt distance on $\partial\widetilde{M} - \{\xi_0\}$. Let $\widetilde{\mu}_e$ be the Patterson–Sullivan measure seen from ξ_0 on $\partial\widetilde{M} - \{\xi_0\}$ (see §2 for the definitions).

THEOREM 1.3. *Assume that $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a map with* $\log f$ *Lipschitz. Let $\widetilde{E}(f)$ be the set of points ξ in $\partial\widetilde{M} - \{\xi_0\}$ such that there exist infinitely many rational lines r with $\widetilde{d}_e(\xi, \widetilde{r}(\infty)) \le f(D(r))e^{-D(r)}$. Then $\widetilde{\mu}_e(\widetilde{E}(f)) = 0$ (respectively $\widetilde{\mu}_e({}^c\widetilde{E}(f)) = 0$) if and only if the integral $\int_1^\infty f(t)^{2(\delta-\delta_0)}\,dt$ converges (respectively diverges).*

If $\widetilde{M} = \mathbb{H}^2, \Gamma = PSL_2(\mathbb{Z})$, then this result is a well-known result of Khintchine (see [**Kh**]). The theorem, and the following corollary, called the *logarithm law* for the geodesic flow, are due to D. Sullivan [**Sul2**] if M has finite volume and constant curvature, to D. Kleinbock and G. Margulis [**KM**] if M is a finite volume locally symmetric space and to B. Stratmann and S. L. Velani [**SV**] if M is geometrically finite with constant curvature.

Endow the unit tangent sphere $T^1_x(M)$ with the Patterson–Sullivan measure (see §2) and denote by $t \mapsto \gamma_v(t)$ the geodesic ray in M defined by $v \in T^1_x(M)$.

THEOREM 1.4. *Assume that $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$. For every x in M and almost every v in $T^1_x(M)$,*

$$\limsup_{t\to+\infty} \frac{d_M(x,\gamma_v(t))}{\log t} = \frac{1}{2(\delta-\delta_0)}.$$

This paper is organized as follows. Section 2 gives the main definitions and notations that will be used in the paper. The main result of the paper, Theorem 1.3, is proved in §5. The main steps are Theorem 1.2 (proved in §3) and a greatly generalized fluctuating density result (see §4). Theorem 1.4 is then proved in §6.

2. *Notation*

2.1. *Generalities.*

This section recalls well-known definitions and results regarding negatively curved metric spaces (see [**Bou, Bow**]).

Let M be a complete pinched negatively curved Riemannian manifold. After normalizing its metric, we assume that its sectional curvature K is normalized by $-\kappa^2 \le K \le -1$ with $1 \le \kappa < +\infty$. Let $\widetilde{\pi} : \widetilde{M} \to M$ be a fixed universal cover, with a covering group Γ.

In particular, $\widetilde{M}$ is a proper geodesic metric space which is CAT(-1) and CAT$_{\text{op}}(-\kappa^2)$, that is its geodesic triangles are more (respectively less) pinched than those in the constant curvature -1 (respectively $-\kappa^2$) space; see [**GH**] for definitions.

Let X be a CAT(-1) geodesic metric space. The *boundary* ∂X of X is the space of all geodesic rays in X, where two rays are identified if they remain within bounded Hausdorff distance. The set $X \cup \partial X$ is endowed with the cone topology.

The *Poincaré series* of a group G of isometries of X is defined by

$$P_G(x,y,s) = \sum_{g\in G} e^{-sd(x,gy)}$$

for any x, y in X and s in $\mathbb{R}_+$. This series converges if $s > \delta_G$ and diverges if $s < \delta_G$ for some $\delta_G \in [0,+\infty]$, which is called the *critical exponent* of G. It is easy to see that δ_G is independent of the points x, y.

Let $a, b \in \partial X$. Their Gromov product with respect to a base point x in X is defined by

$$(a,b)_x = \lim_{t\to+\infty} \tfrac{1}{2}(d(x,a(t)) + d(x,b(t)) - d(a(t),b(t))).$$

It is independent of the geodesic rays $a, b : [0,+\infty[\to X$ representing a, b. The *visual distance d_x* on ∂X is defined by

$$d_x(a,b) = \begin{cases} 0 & \text{if } a = b,\\ e^{-(a,b)_x} & \text{otherwise.}\end{cases}$$

Every isometry γ of X extends to an homeomorphism of ∂X which is an isometry between d_x and $d_{\gamma x}$.

For ξ in ∂X, the *Buseman function* $\beta_\xi : X \times X \to \mathbb{R}$ is defined by

$$\beta_\xi(x,y) = \lim_{t\to\infty} d(x,\xi(t)) - d(y,\xi(t))$$

for any geodesic ray $t \mapsto \xi(t)$ converging to ξ. The *horospheres* centered at ξ are the level sets of $x \mapsto \beta_\xi(x,y)$ (for any $y \in X$), and the *horoballs* are the sublevel sets.

For s in $\mathbb{R}_+$, a *Patterson–Sullivan* (family of) *measure*(s) of dimension s for a group G of isometries of X is a family of absolutely continuous finite measures $(\nu_x)_{x\in X}$ on ∂X such that

(1) $(d\nu_x/d\nu_y)(\xi) = e^{-s\beta_\xi(x,y)}$ for every x, y in X and ξ in ∂X, and

(2) $g_*\nu_x = \nu_{gx}$ for every g in G.

Note that $(\nu_x)_{x\in X}$ is uniquely defined by ν_{x_0}, for x_0 any base point in X.

Given a Patterson–Sullivan family of measures $(\mu_x)_{x\in\widetilde{M}}$ for Γ, for every y in M, the unit tangent sphere $T^1_y M$ can be endowed with a measure, also called a *Patterson–Sullivan measure*, in the following way: take a lift $\widetilde{y}$ of y in $\widetilde{M}$, identify $T^1_y M$ with $T^1_{\widetilde{y}}\widetilde{M}$ by the covering map, and $T^1_{\widetilde{y}}\widetilde{M}$ with $\partial\widetilde{M}$ by the *endpoint map*, which maps a unit vector to the point at infinity of the geodesic ray it defines. The measure $\mu_{\widetilde{y}}$ on $\partial\widetilde{M}$ pulls back to a measure on $T^1_y M$, which by the equivariance property of the Patterson–Sullivan measures does not depend on the chosen lift $\widetilde{y}$.

Assume that X is proper. Let G be a discrete subgroup of isometries of X. The *limit set* ΛG is the set $\overline{Gx}\cap\partial X$, for any x in X. The group G is *non-elementary* if ΛG contains at least three points. If G is non-elementary, the convex hull in X of the limit set of G is denoted by $C\Lambda G$. A point ξ in ΛG is a *conical limit point* of G if it is the endpoint of a geodesic ray in X which projects to a path in $G\backslash X$ that is recurrent in some compact subset. A point ξ in ΛG is a *bounded parabolic point* if it is fixed by some parabolic element in G, and if the quotient $(\Lambda G - \{\xi\})/G_\xi$ is compact, where G_ξ is the stabilizer of ξ. The group G is *geometrically finite* if it is non-elementary and if every limit point of G is conical or bounded parabolic (see [**Bow**] for more information). The manifold M is *non-elementary* or *geometrically finite* if Γ is (as a subgroup of isometries of $\widetilde{M}$).

Assume that the critical exponent δ_G of G satisfies $0 < \delta_G < +\infty$. Assume that G is *of divergent type*, i.e. that the Poincaré series $P_G(x, y, s)$ diverges at $s = \delta_G$. If x_G is a base point in X, the measures ν_x for x in X may then be taken as the weak limit for some $(s_i)_{i\in\mathbb{N}}$ (independent of x) with $s_i > \delta_G$ tending to δ_G as $i \to +\infty$, of

$$\frac{1}{P_G(x_G, x_G, s_i)}\sum_{g\in G} e^{-s_i d(x, g x_G)}\Delta_{g x_G},$$

where Δ_z is a unit Dirac mass at the point z in X.

The *shadow* $\mathcal{O}_x A$ of a subset A of X seen from a point x in $X\cup\partial X$ is the set of points $\xi \neq x$ in ∂X such that the (unique) geodesic ray or line from x to ξ has a non-empty intersection with A.

2.2. *The rational and irrational lines.* The content of this section is taken from [**HP2**], to which we refer for proofs and complements.

Assume that M is non-elementary and has at least one *cusp* e, i.e. an asymptotic class of minimizing geodesic rays in M along which the injectivity radius goes to zero. We say that a geodesic ray *converges* to e if some subray belongs to the class e.

Fix ξ_0 on the boundary $\partial\widetilde{M}$ of $\widetilde{M}$, which is the endpoint of a lift of a geodesic ray converging to e. In particular, ξ_0 is the fixed point of a parabolic element in Γ. Let Γ_0 be the stabilizer of ξ_0 in Γ, called a *parabolic subgroup* for e. We say that the cusp e (and the parabolic subgroup Γ_0) is *bounded* if ξ_0 is a bounded parabolic point for Γ. We denote by $\delta = \delta_\Gamma$ and $\delta_0 = \delta_{\Gamma_0}$ the critical exponents of Γ and Γ_0 respectively. Note that $0 < \delta_0 \leq \delta < +\infty$ (see, for instance, [**Bou**]). If M is compact, then δ is the topological entropy of the geodesic flow of M (see [**Man**]).

Let H_0 be the horosphere centered at ξ_0 such that the horoball HB_0 bounded by H_0 is the maximal horoball centered at ξ_0 such that the quotient of its interior by Γ_0 embeds in M by $\widetilde{\pi}$. Such a maximal horoball exists: see, for instance, [**BuK**]. The subset $\widetilde{\pi}(\mathrm{int}(HB_0))$ of M is called the *maximal Margulis neighborhood* of the cusp e. Fix a base point x_0 in $\widetilde{M}$ belonging to $H_0 \cap C\Lambda\Gamma$.

We define the rational and irrational lines in M and the depth of a rational line as in the introduction. If Γ is geometrically finite, then a geodesic line starting from e in M and contained in $\widetilde{\pi}(C\Lambda\Gamma)$ is rational or irrational or converges to some cusp distinct from e. Any rational line r in M has a lift in $\widetilde{M}$ starting from ξ_0, which is unique modulo the action of Γ_0. The endpoint of any such lift is the center of a horosphere γH_0 for some γ in Γ. The map $r \mapsto \Gamma_0\gamma\Gamma_0$ from the set of rational lines to the set of non-trivial double cosets $\Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0$ is a bijection (see [**HP2**, Lemma 2.7]). It follows from its definition that the depth of r is $d(H_0, \gamma H_0)$. In particular, the number $\mathcal{N}_e(t)$ of rational lines with depth at most t is equal to

$$\mathcal{N}'_e(t) = \mathrm{Card}\{\Gamma_0\gamma\Gamma_0 \in \Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0 \mid d(H_0, \gamma H_0) \leq t\}.$$

If e is a bounded cusp, then since Γ is discrete, the set of depths of rational lines is a discrete subset of $\mathbb{R}$ with finite multiplicities (see [**HP2**]).

We denote by d_{ξ_0} the *Hamenstädt distance* on $\partial\widetilde{M} - \{\xi_0\}$, which is invariant under Γ_0, defined by (see [**HP1**, Appendix], where there is a sign mistake, as well as in [**HP3**])

$$d_{\xi_0}(a, b) = \lim_{t\to+\infty} e^{+t} d_{r(t)}(a, b),$$

with $a, b \in \partial\widetilde{M} - \{\xi_0\}$ and $r : [0, +\infty[\to \widetilde{M}$ a geodesic ray with origin on H_0 and converging to ξ_0. Note that our distance d_{ξ_0} is only equivalent to the distance associated to (ξ_0, H_0) introduced in [**Ham**] but since most of the inspiration comes from this paper, we will, nevertheless, call d_{ξ_0} the Hamenstädt distance.

2.3. *The parabolic manifold.* It turns out that the most intrinsic way of expressing the results mentioned in the introduction is to work with the *parabolic manifold* $M_0 = \Gamma_0\backslash\widetilde{M}$ (see Figure 1).

We denote by H_∞, HB_∞ the image in M_0 (by the canonical map) of H_0, HB_0, respectively. For every rational line r, we denote by H_r, HB_r the image in M_0 of $\gamma H_0, \gamma HB_0$ for any representative γ of the double coset corresponding to r. Note that the subsets HB_r, for the rational lines r, have pairwise disjoint interiors (these are the homeomorphic images of the corresponding open horoballs in $\widetilde{M}$). Note that $D(r) = d_{M_0}(H_\infty, H_r)$ for every rational line r.

We denote by ∞ the point at infinity of M_0 corresponding to the point at infinity ξ_0 of $\widetilde{M}$. Thus when M has finite volume, $HB_\infty = \Gamma_0\backslash HB_0$ is a neighborhood of the end ∞ in M_0. Under the map $\widetilde{M} \to M_0$, the set of orbits under Γ_0 of the geodesic lines starting from ξ_0

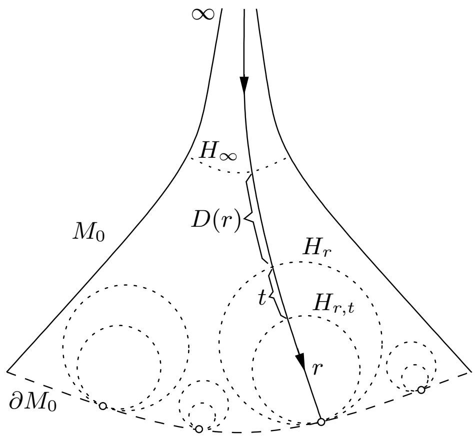

FIGURE 1. The parabolic manifold M_0.

in $\widetilde{M}$ can be (and will be) identified with the set of geodesic lines starting from ∞ in M_0. We denote by ∂M_0 the quotient $\Gamma_0 \backslash (\partial \widetilde{M} - \{\xi_0\})$, which can be (and will be) identified with the set of endpoints of the geodesic lines starting from ∞ in M_0. The endpoint of the rational line r (seen in M_0) is the point at infinity of H_r.

Since the Hamenstädt distance d_{ξ_0} is invariant under Γ_0, we denote by d_∞ its quotient distance on ∂M_0. If L, L' are geodesic lines starting from ∞ in M_0 with endpoints ξ, ξ', we define their *Hamenstädt distance* by $d_\infty(L, L') = d_\infty(\xi, \xi')$.

Let r be a rational line and $t \geq 0$ be given. We denote by $H_{r,t}$ the set of points in HB_r at distance t from H_r. Then $H_{r,t}$ is the image under the projection $\widetilde{M} \to M_0$ of any horosphere γH_t, contained in γHB_0 and at distance t from γH_0, for any representative γ of the double coset corresponding to r.

Let A be a subset of M_0. Define the *shadow* of A seen from ∞ to be the set $\mathcal{O}_\infty A$ of points in ∂M_0 that are the endpoint of a geodesic line starting from ∞ and passing through A. It is the image under the projection $(\partial \widetilde{M} - \{\xi_0\}) \to \partial M_0$ of the shadow seen from ξ_0 of the preimage of A by $\widetilde{M} \to M_0$.

Let $(\mu_x)_{x \in \widetilde{M}}$ be the Patterson–Sullivan measures of dimension δ for Γ, constructed in §2.1 with the basepoint $x_\Gamma = x_0$. Let $p_0 : \partial \widetilde{M} - \{\xi_0\} \to H_0$ be the Γ_0-equivariant homeomorphism, which maps ξ in $\partial \widetilde{M} - \{\xi_0\}$ to the unique intersection point of the horosphere H_0 with the geodesic line between ξ_0 and ξ. Let $\rho : [0, +\infty[\to \widetilde{M}$ be the geodesic ray, with origin x_0 and converging to ξ_0. As t tends to $+\infty$, the measure $e^{\delta t} \mu_{\rho(t)}$ converges weakly to a measure, denoted by μ_{ξ_0}, on $\partial \widetilde{M} - \{\xi_0\}$. Its support is $\Lambda\Gamma - \{\xi_0\}$ and it is invariant by Γ_0. Note that μ_{ξ_0} is absolutely continuous with respect to the Patterson–Sullivan measures; more precisely for every ξ in $\partial \widetilde{M} - \{\xi_0\}$ and $x \in \widetilde{M}$,

$$\frac{d\mu_{\xi_0}}{d\mu_x}(\xi) = e^{-\delta \beta_\xi (p_0(\xi), x)}.$$

By invariance, the measure μ_{ξ_0} induces a measure μ_∞ on ∂M_0, called the *Patterson–Sullivan measure* on ∂M_0. Its support is $\Gamma_0 \backslash (\Lambda\Gamma - \{\xi_0\})$, which is compact if ξ_0 is a bounded parabolic point.

3. *Estimating the relative growth*

We use [**GH**] for notation and background on Gromov-hyperbolic spaces.

THEOREM 3.1. *Let H, G be two discrete subgroups of isometries of a Gromov-hyperbolic proper metric space X, with H contained in G. Let x_0 be any point in X and $f_G(t)$ be the number of g in G such that $d_X(gx_0, x_0) \leq t$. Let $f_{H\backslash G}(t)$ be the number of cosets Hg in $H\backslash G$ such that $d_{H\backslash X}(Hgx_0, Hx_0) \leq t$. If the limit set of H is properly contained in the limit set of G, then there is a constant $c > 0$ such that for every $t > 0$,*

$$\frac{1}{c} f_G(t-c) \leq f_{H\backslash G}(t) \leq f_G(t).$$

Proof. The second inequality is obvious. Let us prove the first one. The result is easy if H is finite; hence, we assume that H is infinite. Note that since ΛH is properly contained in ΛG, the group G is then non-elementary. Recall that a discrete group of isometries of X acts properly on the complement in $X \cup \partial X$ of its limit set.

In particular, there exists a point ξ in ΛG and an open neighbourhood U of ξ in $X \cup \partial X$ such that the number N of elements α in H such that αU meets U is finite. Since ξ belongs to ΛG and G is non-elementary, there exists a hyperbolic element γ in G whose fixed points are both contained in U. Since the action of γ on $X \cup \partial X$ has a North–South dynamics, there exists an integer $N' \geq 0$ such that the sets $\gamma^k U$ for $k = 0, \dots, N'$ cover $X \cup \partial X$.

For $y \in X$ and $V \subset X$, define

$$f_{V,y}(t) = \mathrm{Card}\{g \in G : gx_0 \in B_X(y,t) \cap V\}.$$

Note that $f_{H\backslash G}(t) \geq (1/N) f_{U,x_0}(t)$. For a contradiction, assume that for every integer $n > 0$, there exists $t_n \geq n$ such that $f_{U,x_0}(t_n) \leq (1/n) f_G(t_n - n)$. Let $T = \sup_{k=0,\dots,N'} d(x_0, \gamma^{-k}x_0)$. Then

$$f_G(t_n - T) \leq \sum_{k=0}^{N'} f_{\gamma^k U, x_0}(t_n - T) = \sum_{k=0}^{N'} f_{U,\gamma^{-k}x_0}(t_n - T) \leq N' f_{U,x_0}(t_n) \leq \frac{N'}{n} f_G(t_n - n).$$

Note that f_G is non-decreasing. Hence, for n big enough, say $n \geq \max\{N', T\} + 1$, we have

$$f_G(t_n - T) \leq \frac{N'}{n} f_G(t_n - n) < f_G(t_n - T).$$

This contradiction ends the proof of Theorem 3.1. □

The preceding theorem (whose proof was inspired by [**Rob**]) and the following easy and well-known result imply Theorem 1.1 of the introduction.

LEMMA 3.2. *Let H, G be two non-elementary quasi-convex discrete subgroups of isometries of a Gromov-hyperbolic proper metric space X, with H contained in G. Then H has finite index in G if and only if the limit set of H equals the limit set of G.* □

We now turn to the estimation of the growth of $\mathcal{N}_e(t)$, which is one of the main steps in the proof of the Khintchine–Sullivan Theorem 5.1. The following lemma is obvious.

LEMMA 3.3. *For every positive constants A, δ, δ'_0 with $\delta > \delta'_0$, there exist an integer $N \geq 1$ and a constant $B > 0$ such that for all positive real sequences $(b_n), (c_n)$ with $c_n \leq Ae^{\delta'_0 n}$, $b_n \leq Ae^{\delta n}$ and $\sum_{k=0}^{n} b_k c_{n-k} \geq (1/A)e^{\delta n}$, we have $\sum_{k=1}^{N} b_{n+k} \geq Be^{\delta n}$.*

Proof. Let

$$N = E\left(\frac{1}{\delta - \delta'_0}\left|\log \frac{A^3}{1 - e^{\delta'_0 - \delta}}\right|\right) + 2,$$

where $E(\cdot)$ denotes the integer part. Note that

$$\sum_{k=0}^{n} b_k c_{n+N-k} \leq \sum_{k=0}^{n} Ae^{\delta k} Ae^{\delta'_0(n+N-k)} = A^2 e^{\delta'_0(n+N)} \frac{e^{(\delta-\delta'_0)(n+1)} - 1}{e^{\delta-\delta'_0} - 1} \leq \frac{A^2 e^{\delta'_0 N}}{1 - e^{\delta'_0 - \delta}} e^{\delta n}.$$

Since $c_{n+N-k} \leq Ae^{\delta'_0 N}$ for $k \geq n+1$, we have

$$\begin{aligned} Ae^{\delta'_0 N} \sum_{k=n+1}^{n+N} b_k \geq \sum_{k=n+1}^{n+N} b_k c_{n+N-k} &= \sum_{k=0}^{n+N} b_k c_{n+N-k} - \sum_{k=0}^{n} b_k c_{n+N-k} \\ &\geq \left(\frac{1}{A} e^{\delta N} - \frac{A^2 e^{\delta'_0 N}}{1 - e^{\delta'_0 - \delta}}\right) e^{\delta n}, \end{aligned}$$

which proves the result, by the definition of N. □

THEOREM 3.4. *With the notation in §2, assume that e is a bounded cusp. If $f_{\widetilde{\pi}}(t) \asymp e^{\delta t}$ and $\delta_0 < \delta$, then $\mathcal{N}'_e(t) \asymp e^{\delta t}$.*

Proof. Let $X = C\Lambda\Gamma$, which is a convex subset of $\widetilde{M}$, and, hence, an η-hyperbolic metric space for some $\eta \geq 0$. Recall that X is Γ-invariant and closed and contains x_0.

Choose a representative for every non-trivial double coset $[\gamma]$ in $\Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0$ and denote it by the same symbol, such that

$$d(x_0, [\gamma]x_0) = \min_{\alpha,\alpha' \in \Gamma_0} d(x_0, \alpha[\gamma]\alpha' x_0).$$

LEMMA 3.5. *The geodesic segment $[x_0, [\gamma]x_0]$ is at bounded distance from the common perpendicular segment between HB_0 and $[\gamma]HB_0$.*

Proof. For every γ in $\Gamma - \Gamma_0$, the common perpendicular segment $[u, v]$ in $\widetilde{M}$ between HB_0 and γHB_0 (with $u \in HB_0$) is contained in X. Its length is the minimal distance between a point in HB_0 and a point in γHB_0. Since $\Gamma_0\backslash(H_0 \cap X)$ is compact, by multiplying γ on the left (respectively right) by an element of Γ_0, the distance $d(x_0, u)$ (respectively $d(\gamma x_0, v)$) can be made less than a constant. The result follows. □

By this lemma, there exists a constant $\tau_1 \geq 0$ such that

$$d(x_0, [\gamma]x_0) - \tau_1 \leq d(H_0, [\gamma]H_0) \leq d(x_0, [\gamma]x_0).$$

Since $f_{\widetilde{\pi}}(t) \leq ce^{\delta t}$ for some $c > 0$, we immediately have the upper bound $\mathcal{N}'_e(t) \leq ce^{\delta\tau_1}e^{\delta t}$. Let us now prove the analogous minoration.

The horoballs HB_0 and $[\gamma]HB_0$ are convex. Then, by Lemma 3.5, the piecewise geodesic path from $\alpha^{-1}x_0$ to x_0, then from x_0 to $[\gamma]x_0$, then from $[\gamma]x_0$ to $[\gamma]\alpha' x_0$, is quasigeodesic in X. Therefore, there exists an integer τ_2, such that for every $[\gamma]$ in $\Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0$ and every α, α' in Γ_0,

$$d(x_0, \alpha[\gamma]\alpha' x_0) - \tau_2 \leq d(x_0, \alpha x_0) + d(H_0, [\gamma]H_0) + d(x_0, \alpha' x_0) \leq d(x_0, \alpha[\gamma]\alpha' x_0) + \tau_2. \quad (*)$$

Since $f_{\widetilde{\pi}}(t) \geq ce^{\delta t}$ for some $c > 0$ and by Lemma 3.3 (with $c_n = 1$ for all n), there exist an integer N and a constant $\tau_3 > 0$ such that $a_n \geq \tau_3 e^{\delta n}$, where

$$a_n = \mathrm{Card}\{\gamma \in \Gamma \mid n - N < d(x_0, \gamma x_0) \leq n\}.$$

Up to normalizing the metric of X by $1/N$, we may (and we will) assume that $N = 1$. Indeed, let Y be a proper η-hyperbolic space. Let Γ be a discrete group of isometries of Y with a critical exponent δ. Let $\epsilon > 0$ be a given constant. Then the metric space ϵY (which is the set Y with the metric $d_{\epsilon Y} = \epsilon d_Y$) is a proper $(\epsilon\eta)$-hyperbolic space and the group Γ is still a discrete group of isometries of ϵY, with critical exponent δ/ϵ. It follows easily that if we prove the result for $(1/N)X$, then it also holds for X.

Let δ_0' be a real number such that $\delta_0 < \delta_0' < \delta$. In particular, by the definition of δ_0, we have $\mathrm{Card}\{\alpha \in \Gamma_0 \mid d(x_0, \alpha x_0) \leq n\} = O(e^{\delta_0' n})$. Let

$$b_k = \mathrm{Card}\{([\gamma], \alpha') \in (\Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0) \times \Gamma_0 \mid k - 1 < d(H_0, [\gamma]H_0) + d(x_0, \alpha' x_0) \leq k\}$$

and

$$c_k = \mathrm{Card}\{\alpha \in \Gamma_0 \mid k - 1 - 2\tau_2 < d(x_0, \alpha x_0) \leq k + 1\}.$$

By the formula $(*)$, we have

$$a_n \leq \sum_{k=0}^{n+\tau_2} b_k c_{n+\tau_2-k} + \mathrm{Card}\{\alpha \in \Gamma_0 \mid n - 1 < d(x_0, \alpha x_0) \leq n\}.$$

Since the last cardinal is $O(e^{\delta_0' n})$, the assumptions of Lemma 3.3 for the sequences $(b_n), (c_n)$ are satisfied for some constant $A > 0$. Hence, there exist an integer N' and a constant $c' > 0$ such that $\mathrm{Card}\{([\gamma], \alpha') \in (\Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0) \times \Gamma_0 \mid n - N' < d(H_0, [\gamma]H_0) + d(x_0, \alpha' x_0) \leq n\} \geq c' e^{\delta n}$.

Iterating this procedure, we get the minoration. □

Theorem 1.2 in the introduction now follows from Theorem 3.4 and the following lemma.

LEMMA 3.6. *If M is geometrically finite and if the critical exponent of each parabolic group is strictly less than δ, then $f_{\widetilde{\pi}}(t) \asymp e^{\delta t}$.*

Proof. Since Γ contains a parabolic element, the length spectrum of M is non-discrete in $\mathbb{R}$ (see [**Dal**]). Since M is geometrically finite and the critical exponent of each parabolic group is strictly less than δ, it follows from [**DOP**] that the Bowen–Margulis measure of M is finite. The result then follows by [**Rob**], as recalled in the introduction. □

4. *The fluctuating density property*

This section is devoted to the proof of the following result, which is the second key step in the proof of the Khintchine–Sullivan Theorem 5.1. It is basically due to Sullivan [**Sul3**] in the finite volume, constant curvature case and due to Stratmann and Velani [**SV**] in the geometrically finite, constant curvature case. See also [**HV**] for other applications.

THEOREM 4.1. *With the notation from §2, assume that e is a bounded cusp, Γ is of divergent type and $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$. Then there exists a constant $c > 0$ such that for every rational line r and every $t \geq 0$, one has*

$$\frac{1}{c} e^{-\delta D(r) + 2(\delta_0 - \delta)t} \leq \mu_\infty(\mathcal{O}_\infty H_{r,t}) \leq c e^{-\delta D(r) + 2(\delta_0 - \delta)t}.$$

Remarks. (1) The assumption that Γ is of divergent type is sufficient for the application to Theorem 5.1. However, it can be removed from the statement, by using in the proof of Proposition 4.2 the general construction of the Patterson–Sullivan measure. Note that if $f_{\widetilde{\pi}}(t) \asymp e^{\delta t}$, then Γ is of divergent type.

(2) The assumption that $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$ cannot be removed, since some control of the Poincaré series of the parabolic subgroup is needed in order to estimate the behaviour inside horoballs of the Patterson–Sullivan measures. Note that it implies that Γ_0 is of divergent type, hence by [**DOP**, Proposition 2] that $\delta_0 < \delta$.

Let us consider the map $\phi : \widetilde{M} \to \mathbb{R}$, where $\phi(y)$ is the total mass of the Patterson–Sullivan measure μ_y. Note that, by the equivariance properties of the Patterson–Sullivan measures, we have $\phi(\gamma y) = \phi(y)$ for every y in $\widetilde{M}$ and γ in Γ.

PROPOSITION 4.2. *Assume that Γ is of divergent type and $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$. For every $A \geq 0$, there exists $B > 0$ such that for every y in $\widetilde{M}$ at distance at most A from the geodesic ray from x_0 to ξ_0, we have*

$$\frac{1}{B} e^{(2\delta_0 - \delta) d(x_0, y)} \leq \phi(y) \leq B e^{(2\delta_0 - \delta) d(x_0, y)}.$$

Proof. Let $P = P_\Gamma$ be the Poincaré series of Γ. As Γ is of divergent type, and as the constant map with value 1 on $\widetilde{M} \cup \partial \widetilde{M}$ is continuous with compact support, it follows from the construction of the Patterson–Sullivan measures (see §2) that, for some $s_i \to \delta^+$,

$$\phi(y) = \lim_{i \to +\infty} \frac{P(y, x_0, s_i)}{P(x_0, x_0, s_i)}.$$

Choose a representative for every right coset $[\gamma]$ in $\Gamma_0 \backslash \Gamma$ and denote it by the same symbol, such that

$$d(x_0, [\gamma] x_0) = \min_{\alpha \in \Gamma_0} d(x_0, \alpha[\gamma] x_0).$$

By the properties of quasi-geodesics in the Gromov-hyperbolic space $\widetilde{M}$, for every $A \geq 0$, there exists a constant $c_1 > 0$, depending only on A, such that if y is at distance at most A from the geodesic between x_0 and ξ_0, then

$$\begin{aligned} d(y, \alpha y) + d(y, x_0) + d(x_0, [\gamma] x_0) - c_1 &\leq d(y, \alpha[\gamma] x_0) \\ &\leq d(y, \alpha y) + d(y, x_0) + d(x_0, [\gamma] x_0). \end{aligned}$$

Let $Q(x_0, x_0, s) = \sum_{[\gamma]\in\Gamma_0\backslash\Gamma} e^{-sd(x_0,[\gamma]x_0)}$. By uniquely writing each element of Γ in the form $\alpha[\gamma]$ for $\alpha \in \Gamma_0$, we get from these inequalities that, for $s > \delta$,

$$e^{-sd(y,x_0)}P_{\Gamma_0}(y, y, s)Q(x_0, x_0, s) \le P(y, x_0, s) \le e^{sc_1}e^{-sd(y,x_0)}P_{\Gamma_0}(y, y, s)Q(x_0, x_0, s).$$

Hence, as the series $P_{\Gamma_0}(y, y, s)$ converges at $s = \delta$,

$$e^{-\delta c_1}e^{-\delta d(y,x_0)}\frac{P_{\Gamma_0}(y, y, \delta)}{P_{\Gamma_0}(x_0, x_0, \delta)} \le \phi(y) \le e^{\delta c_1}e^{-\delta d(y,x_0)}\frac{P_{\Gamma_0}(y, y, \delta)}{P_{\Gamma_0}(x_0, x_0, \delta)}. \tag{i}$$

Recall that y is at distance at most A from the geodesic from x_0 to ξ_0. Hence, there exists a constant c_2 depending only on A such that, for every α in Γ_0, if $d(y, \alpha y) \ge c_2$, then the piecewise geodesic from x_0 to y, then from y to αy, then from αy to αx_0 is a quasi-geodesic. Therefore, there exists a constant $c_3 \ge 0$ such that, if $d(y, \alpha y) \ge c_2$, then

$$d(x_0, \alpha x_0) \ge 2d(x_0, y) + d(y, \alpha y) - c_3 \ge 2d(x_0, y) + c_2 - c_3.$$

Note that, by the triangular inequality, $d(x_0, \alpha x_0) \le 2d(x_0, y) + d(y, \alpha y)$. In particular, if $d(y, \alpha y) < c_2$, then $d(x_0, \alpha x_0) < 2d(x_0, y) + c_2$.

Hence,

$$\begin{aligned} P_{\Gamma_0}(y, y, \delta) &= \sum_{\substack{\alpha\in\Gamma_0\\ d(y,\alpha y)<c_2}} e^{-\delta d(y,\alpha y)} + \sum_{\substack{\alpha\in\Gamma_0\\ d(y,\alpha y)\ge c_2}} e^{-\delta d(y,\alpha y)} \\ &\le \mathrm{Card}\{\alpha \in \Gamma_0 : d(x_0, \alpha x_0) \le 2d(y, x_0) + c_2\} \\ &\quad + e^{2\delta d(y,x_0)} \sum_{\substack{\alpha\in\Gamma_0\\ d(x_0,\alpha x_0)\ge 2d(y,x_0)+c_2-c_3}} e^{-\delta d(x_0,\alpha x_0)}. \end{aligned}$$

The last sum is at most

$$\sum_{\substack{n\in\mathbb{N}\\ n\ge 2d(y,x_0)+c_2-c_3-1}} \mathrm{Card}\{\alpha \in \Gamma_0 : n \le d(x_0, \alpha x_0) \le n + 1\}e^{-\delta n}.$$

Recall that, by assumption, there exists a constant $c_4 > 0$ such that

$$\frac{1}{c_4}e^{\delta_0 n} \le \mathrm{Card}\{\alpha \in \Gamma_0 : d(x_0, \alpha x_0) \le n\} \le c_4 e^{\delta_0 n}.$$

Hence

$$\begin{aligned} P_{\Gamma_0}(y, y, \delta) &\le c_4 e^{\delta_0(2d(y,x_0)+c_2)} + e^{2\delta d(y,x_0)} \sum_{\substack{n\in\mathbb{N}\\ n\ge 2d(y,x_0)+c_2-c_3-1}} c_4 e^{\delta_0(n+1)}e^{-\delta n} \\ &\le c_4 e^{\delta_0(2d(y,x_0)+c_2)} + c_4 e^{\delta_0} e^{2\delta d(y,x_0)} \frac{e^{(\delta_0-\delta)(2d(y,x_0)+c_2-c_3-1)}}{1 - e^{\delta_0-\delta}} = c_5 e^{2\delta_0 d(y,x_0)}, \end{aligned}$$

for some constant $c_5 > 0$ depending only on $\delta, \delta_0, c_2, c_3, c_4$.

Conversely,

$$\begin{aligned} P_{\Gamma_0}(y, y, \delta) &\ge \sum_{\substack{\alpha\in\Gamma_0\\ d(y,\alpha y)<c_2}} e^{-\delta d(y,\alpha y)} \\ &\ge e^{-\delta c_2}\mathrm{Card}\{\alpha \in \Gamma_0 : d(x_0, \alpha x_0) < 2d(y, x_0) + c_2 - c_3\} \\ &\ge \frac{1}{c_4}e^{-\delta c_2}e^{\delta_0(2d(y,x_0)+c_2-c_3-1)} = c_6 e^{2\delta_0 d(y,x_0)}, \end{aligned}$$

for some constant $c_6 > 0$ depending only on $\delta, \delta_0, c_2, c_3, c_4$.

Since $P_{\Gamma_0}(x_0, x_0, \delta)$ does not depend on y, Proposition 4.2 follows from equation (i) and the lower and upper bounds on $P_{\Gamma_0}(y, y, \delta)$. □

For every rational line r, let γ_r be any representative of the double class in $\Gamma_0\backslash(\Gamma - \Gamma_0)/\Gamma_0$ corresponding to r such that

$$d(x_0, \gamma_r x_0) = \min_{\alpha,\beta\in\Gamma_0} d(x_0, \alpha\gamma_r\beta x_0).$$

PROPOSITION 4.3. *With the notation from §2, assume that e is a bounded cusp. There exists a constant $c > 0$ such that for every $t \geq 0$ and every rational ray r, if $y_{r,t}$ is the intersection point of $\gamma_r H_t$ and the geodesic ray from $\gamma_r x_0$ to $\gamma_r \xi_0$, then*

$$\frac{1}{c} e^{-\delta(t+D(r))}\phi(y_{r,t}) \leq \mu_\infty(\mathcal{O}_\infty H_{r,t}) \leq c e^{-\delta(t+D(r))}\phi(y_{r,t}).$$

Proof. By the definition of μ_∞ and $\mathcal{O}_\infty$ (see §2), we have

$$\mu_\infty(\mathcal{O}_\infty H_{r,t}) = \mu_{\xi_0}(\mathcal{O}_{\xi_0}\gamma_r H_t).$$

We start with a few preliminary remarks. By Lemma 3.5, there is a constant $c_7 \geq 0$ such that

$$d(H_0, \gamma_r H_0) \leq d(x_0, \gamma_r x_0) \leq d(H_0, \gamma_r H_0) + c_7.$$

Recall from §2 that $D(r) = d(H_0, \gamma_r H_0)$. The point $\gamma_r x_0$ lies at a uniformly (in r) bounded distance from the geodesic ray between x_0 to $\gamma_r \xi_0$, with its orthogonal projection to this ray lying between x_0 and the orthogonal projection to this ray of $y_{r,t}$, for t big enough.

Hence, there exists a constant $c_8 \geq 0$ such that for every rational line r and $t \geq 0$,

$$|d(x_0, y_{r,t}) - D(r) - t| \leq c_8.$$

By the properties of the measure μ_{ξ_0} (see §2.3), for every compact subset K of $\partial\widetilde{M} - \{\xi_0\}$, there exists a constant $c_9 > 0$ such that, for every ξ in K,

$$\frac{1}{c_9} \leq \frac{d\mu_{\xi_0}}{d\mu_{x_0}}(\xi) \leq c_9.$$

In what follows, K will be any compact subset of $\partial\widetilde{M} - \{\xi_0\}$ containing the shadows seen from ξ_0 of every horoball $\gamma_r H_0$ as r ranges over the rational lines. Such a K exists, since by the choice of the representatives $[\gamma]$, there exists $R > 0$ such that $\mathcal{O}_{\xi_0}\gamma_r H_0 \subset \mathcal{O}_{\xi_0} B(x_0, R)$ for every rational line r.

Step 1. Let us prove first the upper bound in Proposition 4.3.

By the properties of the Patterson–Sullivan measures (see §2.1), we have, with β the Buseman function for $\widetilde{M}$,

$$\phi(y_{r,t}) = \int_{\partial\widetilde{M}} d\mu_{y_{r,t}}(\xi) \geq \int_{\mathcal{O}_{\xi_0}\gamma_r H_t} d\mu_{y_{r,t}}(\xi) = \int_{\mathcal{O}_{\xi_0}\gamma_r H_t} e^{-\delta\beta_\xi(y_{r,t},x_0)}\, d\mu_{x_0}(\xi).$$

Recall that x_0 (respectively $y_{r,t}$) are at uniformly (in r, t) bounded distances from the intersection point with H_0 (respectively $\gamma_r H_t$) of the geodesic line between ξ_0 and $\gamma_r \xi_0$. Hence, there exists a constant $c_{10} \geq 0$ such that for every ξ in $\mathcal{O}_{\xi_0}\gamma_r H_t$,

$$\beta_\xi(y_{r,t}, x_0) \leq -d(y_{r,t}, x_0) + c_{10}.$$

Hence,

$$\phi(y_{r,t}) \geq e^{-\delta c_{10}} e^{\delta d(x_0, y_{r,t})} \int_{\mathcal{O}_{\xi_0}\gamma_r H_t} d\mu_{x_0}(\xi) \geq e^{-\delta c_{10}} e^{-\delta c_8} e^{\delta(D(r)+t)} \mu_{x_0}(\mathcal{O}_{\xi_0}\gamma_r H_t)$$
$$\geq \frac{1}{c_9} e^{-\delta(c_{10}+c_8)} e^{\delta(D(r)+t)} \mu_{\xi_0}(\mathcal{O}_{\xi_0}\gamma_r H_t).$$

This proves the first step.

Step 2. Let us now prove the lower bound in Proposition 4.3.

For a contradiction, suppose that there exist a sequence of rational lines $(r_i)_{i\in\mathbb{N}}$ and a sequence of non-negative real numbers $(t_i)_{i\in\mathbb{N}}$ with t_i tending to $+\infty$, such that, with $y_i = y_{r_i,t_i}$ and $H_i = \gamma_{r_i} H_{t_i}$,

$$\frac{1}{\phi(y_i)} e^{\delta(t_i+D(r_i))} \mu_{\xi_0}(\mathcal{O}_{\xi_0} H_i)$$

tends to zero as i tends to ∞.

Let $(X_i, *_i, d_i, G_i)_{i\in\mathbb{N}}$ be a sequence of pointed metric spaces with group of isometries, where $X_i = \widetilde{M}, *_i = y_i, d_i = d, G_i = \Gamma$. Since $\widetilde{M}$ has pinched negative curvature $-\kappa^2 \leq K \leq -1$, up to extracting a subsequence, the sequence $(X_i, *_i, d_i, G_i)_{i\in\mathbb{N}}$ converges for the equivariant pointed Hausdorff–Gromov convergence (see [**Fuk**]) to a proper CAT(-1) and CAT$_{\rm op}(-\kappa^2)$ pointed geodesic metric space with group of isometries, that we denote by $(X_\infty, *_\infty, d_\infty, G_\infty)$. In particular, the metric spaces $(\partial X_i, d_{*_i})$ converge for the Hausdorff–Gromov convergence to $(\partial X_\infty, d_{*_\infty})$. We fix a definite convergence $(X_i, *_i) \to_i (X_\infty, *_\infty)$ (see [**Gro1**]), which induces a definite convergence $\partial X_i \to_i \partial X_\infty$.

Let $\nu_i = (1/\phi(y_i))\mu_{y_i}$, which is a probability measure on ∂X_i. Up to extracting a subsequence, the metric measured spaces $(\partial X_i, d_{*_i}, \nu_i)$ converge to the metric measured space $(\partial X_\infty, d_{*_\infty}, \nu_\infty)$; see [**Gro2**, ch. $3\frac{1}{2}$]. We may assume that if $f_i : \partial X_i \to \mathbb{R}$ are continuous maps converging to a continuous map $f : \partial X_\infty \to \mathbb{R}$ for the definite convergence $\partial X_i \to \partial X_\infty$, then $\nu_\infty(f) = \lim_{i\to\infty} \nu_i(f_i)$.

Since the horoball H_0 is precisely invariant under Γ and t_i tends to $+\infty$, for i big enough, the only elements in G_i which move the point $*_i$ less than any given constant are parabolic. By taking iterates, there exists $0 < a \leq b < +\infty$ and, for every i in $\mathbb{N}$, some α_i in G_i such that $a \leq d_i(\alpha_i *_i, *_i) \leq b$. Hence, G_∞ is a non-trivial parabolic group of isometries, fixing the point $\xi_\infty = \lim_i \gamma_{r_i}\xi_0$ of ∂X_∞. Since ν_i is a Patterson–Sullivan measure of dimension $\delta_i = \delta$ for G_i, the measure ν_∞ is a Patterson–Sullivan measure of dimension δ for the isometry group G_∞ on X_∞. Since G_∞ is parabolic and non-trivial, any closed subset of ∂X_∞ not containing ξ_∞ may be sent into any neighbourhood of ξ_∞ by some element of G_∞. By the absolutely continuous property, the measure for ν_∞ of any neighbourhood of ξ_∞ is, therefore, non-zero.

Recall that for every ξ in $\partial\widetilde{M}$ and every u, v in $\widetilde{M}$, one has, by the triangle inequality, $\beta_\xi(u, v) \geq -d(u, v)$. By the properties of the Patterson–Sullivan measures, we have

$$\nu_i(\mathcal{O}_{\xi_0} H_i) = \frac{1}{\phi(y_i)} \mu_{y_i}(\mathcal{O}_{\xi_0} H_i) \leq \frac{1}{\phi(y_i)} e^{\delta d(y_i, x_0)} \mu_{x_0}(\mathcal{O}_{\xi_0} H_i)$$
$$\leq e^{\delta c_8} \frac{1}{\phi(y_i)} e^{\delta(t_i+D(r_i))} \mu_{x_0}(\mathcal{O}_{\xi_0} H_i) \leq c_9 e^{\delta c_8} \frac{1}{\phi(y_i)} e^{\delta(t_i+D(r_i))} \mu_{\xi_0}(\mathcal{O}_{\xi_0} H_i).$$

Hence, $\nu_i(\mathcal{O}_{\xi_0}H_i)$ tends to zero as i tends to $+\infty$. The family of subsets $\mathcal{O}_{\xi_0}H_i$ converges for the definite convergence $\partial X_i \to \partial X_\infty$, up to extracting a subsequence, to a subset V of ∂X_∞ which is a neighbourhood of ξ_∞. Indeed, up to extracting a subsequence, the points $\xi_0 \in \partial X_i$ converge to $\xi_{0,\infty} \in \partial X_\infty$; since the horoball H_i is centered at $\gamma_{r_i}\xi_0$ and passes through y_i, the point $*_\infty$, which is the limit of $y_i = *_i$, belongs to the geodesic between $\xi_{0,\infty}$ and ξ_∞, and V is the shadow seen from $\xi_{0,\infty}$ of the horosphere centered at ξ_∞ and passing through $*_\infty$. Let U be an open neighbourhood of ξ_∞ such that $\overline{U} \subset \mathring{V}$. Let f be a continuous map, with support contained in U, bounded by 1, with value 1 in some neighbourhood of ξ_∞. It is easy to construct continuous maps $f_i : \partial X_i \to \mathbb{R}$ with support in $\mathcal{O}_{\xi_0}H_i$, bounded by 1, which converge to f under $\partial X_i \to \partial X_\infty$. Since $\nu_i(f_i) \leq \nu_i(\mathcal{O}_{\xi_0}H_i)$, it follows that $\nu_\infty(f) = \lim_{i\to\infty}\nu_i(f_i) = 0$. This contradicts the fact that ξ_∞ belongs to the support of the measure ν_∞. □

Proof of Theorem 4.1. Note that $\gamma_r^{-1}y_{r,t}$ is the point at distance t from x_0 on the geodesic between x_0 and ξ_0. Hence, by the invariance of ϕ and by Proposition 4.2, one has $\phi(y_{r,t}) = \phi(\gamma_r^{-1}y_{r,t}) \asymp e^{(2\delta_0-\delta)t}$. The result then follows from Proposition 4.3. □

5. *The Khintchine–Sullivan theorem in variable curvature*

A map $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *slowly varying* if it is measurable and if there exist constants $B > 0$ and $A \geq 1$ such that for every x, y in $\mathbb{R}_+$, if $|x - y| \leq B$, then $f(y) \leq Af(x)$. This implies, in particular, that f is locally bounded, hence locally integrable. Note that f is slowly varying if and only if there is a constant $C \geq 1$ such that for every x, y in $\mathbb{R}_+$, if $|x - y| \leq 1$, then $|\log f(x) - \log f(y)| \leq C$. In particular, if $\log f$ is Lipschitz, then f is slowly varying. If f is slowly varying, with C as before, then for every $\epsilon > 0$ and $N \in \mathbb{N}$,

$$e^{-CN\epsilon}\sum_{n=1}^{\infty} f(Nn)^\epsilon \leq \int_N^\infty f(t)^\epsilon\, dt \leq e^{CN\epsilon}\sum_{n=1}^{\infty} f(Nn)^\epsilon.$$

Let d_∞ be the Hamenstädt distance on the set of geodesic lines starting from ∞ in M_0 (see §2.3).

THEOREM 5.1. *With the notation of §2, assume that e is a bounded cusp, $f_{\widetilde{\pi}}(t) \asymp e^{\delta t}$, and $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be slowly varying. Let $E(f)$ be the set of geodesic lines ξ in M_0 starting from ∞ such that there exist infinitely many rational lines r in M_0 with $d_\infty(\xi, r) \leq f(D(r))e^{-D(r)}$. Then $\mu_\infty(E(f)) = 0$ if and only if the integral $\int_1^\infty f(t)^{2(\delta-\delta_0)}\,dt$ converges and $\mu_\infty({}^cE(f)) = 0$ if and only if the integral $\int_1^\infty f(t)^{2(\delta-\delta_0)}\,dt$ diverges.*

Note that Theorem 1.3 in the introduction then follows from Lemma 3.6. By the remarks following Theorem 4.1, the assumptions of Theorem 5.1 imply that Γ is of divergent type and that $\delta_0 < \delta$. We start the proof of Theorem 5.1 by some reduction on f.

LEMMA 5.2. *For every constant $\eta > 0$, to prove this theorem, it is sufficient to prove it when, furthermore, $f(t) \leq \eta$ for every t in $\mathbb{R}_+$.*

Proof. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be slowly varying. For $\eta > 0$, let $f' = \inf\{\eta, f\}$, which is also slowly varying. Assume that Theorem 5.1 holds for f'. Let us prove that it holds for f. Let F be the set of t in $\mathbb{R}_+$ such that $f(t) > \eta$.

If F is bounded, then $E(f) = E(f')$ since there are only finitely many rational lines r with $D(r)$ less than a constant. The convergence of the integral of $f^{2(\delta-\delta_0)}$ does not depend on the values of $f(t)$ for t less than a constant. Hence, Theorem 5.1 holds for f if and only if it holds for f'.

Assume that F is unbounded. Since f is slowly varying, the integral of $f^{2(\delta-\delta_0)}$ diverges, as well as the integral of $(f')^{2(\delta-\delta_0)}$. Note that $E(f') \subset E(f)$. If the theorem holds for f', then $\mu_\infty({}^c E(f')) = 0$. Hence, $\mu_\infty({}^c E(f)) = 0$, so that the theorem holds for f. □

In particular, we assume from now on that $f(t) \leq 1$.

By Theorem 3.4 and Lemma 3.3, there exist $c_1' > 0$ and an integer $N \geq 1$ such that for every n in $\mathbb{N}$, the number $\mathcal{N}_e''(n)$ of rational lines r such that $n \leq D(r) < n + N$ satisfies

$$\frac{1}{c_1'} e^{\delta n} \leq \mathcal{N}_e''(n) \leq c_1' e^{\delta n}.$$

Define $H_{r,f} = H_{r,-\log f \circ D(r)}$ with the notation of §2.3. Let $\mathcal{A}_n$ be the set of shadows seen from ∞ of the $H_{r,f}$'s where r ranges over the rational lines with $Nn \leq D(r) < (n+1)N$. Define $A_n = \cup \mathcal{A}_n$, which is a subset of ∂M_0. The proof of Theorem 5.1 is based on the next two propositions.

PROPOSITION 5.3. *The sum* $\sum_{n=0}^{\infty} \mu_\infty(A_n)$ *diverges if and only if the integral* $\int_1^\infty f^{2(\delta-\delta_0)}$ *diverges.*

Proof. We start with the following lemma.

LEMMA 5.4. *For every* $A \geq 0$*, there exists* $B \geq 0$ *such that the following holds. Let* X *be a CAT(−1) space, and* ξ_0, ξ_1, ξ_2 *be distinct points at infinity of* X*. Let* H_i *for* $i = 1, 2$ *be horospheres centered at* ξ_i *respectively, bounding disjoint open horoballs. Let* x_i *be the intersection point with* H_i *of the geodesic line between* ξ_0 *and* ξ_i*. For* $t \geq 0$*, let* $H_{i,t}$ *be the horosphere centered at* ξ_i*, contained in the horoball bounded by* H_i *and at distance* t *from* H_i*. If* $|\beta_{\xi_0}(x_1, x_2)| \leq A$*, then the shadows seen from* ξ_0 *of* $H_{1,B}$ *and* $H_{2,B}$ *are disjoint.*

Proof. By the techniques of approximation by trees (see [**GH**, p. 33] or [**CDP**, Ch. 8]), this lemma follows from the particular case when X is a tree T (though the constant B might be worse). See Figure 2.

Let $B = A/2 + 1$. As a preliminary remark, note that if ξ, ξ' are distinct ends of the tree T, if H is a horosphere centered at ξ' and x is the intersection point with H of the geodesic line between ξ and ξ', then $\mathcal{O}_\xi H = \mathcal{O}_\xi x$, since any geodesic line starting from ξ that meets H has to go through x.

With the notation of the claim, we consider two cases. Either x_1 belongs to the geodesic between ξ_0 and x_2, or it does not. In the second case (assuming that x_2 does not belong to the geodesic ray between x_1 and ξ_0, otherwise the situation is symmetric to the first case), the shadows (seen from ξ_0) of x_1 and of x_2 are disjoint, hence the shadows

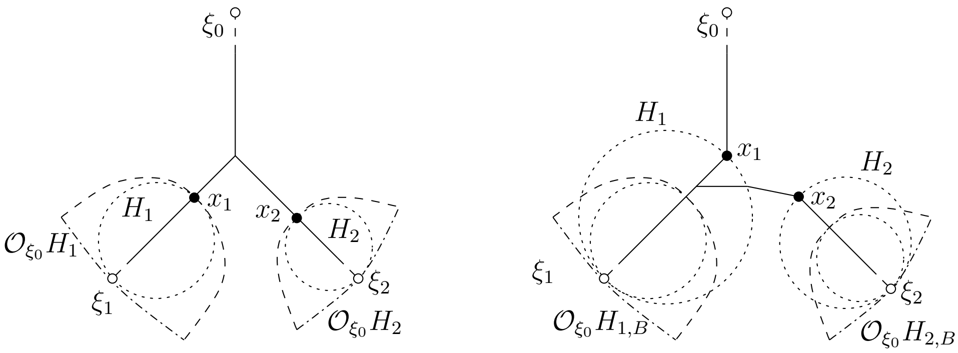

FIGURE 2. Separating shadows in trees.

of $H_{1,t}$, $H_{2,t}$ are disjoint for any $t \geq 0$. Assume that the first case holds. In particular, $d(x_1, x_2) = |\beta_{\xi_0}(x_1, x_2)| \leq A$. (Though we will not need it, note that the shadow of H_2 is, hence, contained in the shadow of H_1.) Since H_1 and H_2 bound disjoint open horoballs, the point x_2 does not lie in the open horoball bounded by H_1. Hence, the intersection of $[x_1, x_2]$ with the geodesic ray between x_1 and ξ_1 has length at most $A/2$. It follows that the shadows seen from ξ_0 of $H_{1,B}$ and $H_{2,B}$ are disjoint. □

We now prove Proposition 5.3. By Lemma 5.4, there exists a constant $c'_2 > 0$ (depending only on N) such that for every n in $\mathbb{N}$ and all distinct rational lines r, r' with $Nn \leq D(r), D(r') < N(n+1)$, the intersection of $\mathcal{O}_\infty H_{r,c'_2}$ and $\mathcal{O}_\infty H_{r',c'_2}$ is empty. By the reduction argument on f, we assume from now on that $f(t) \leq e^{-c'_2}$ for every t. In particular, $\mathcal{O}_\infty H_{r,f}$ is contained in $\mathcal{O}_\infty H_{r,c'_2}$. Hence, the union $A_n = \cup \mathcal{A}_n$ is a disjoint union. By Theorem 4.1, we then have

$$\mu_\infty(A_n) = \sum_{Nn \leq D(r) < N(n+1)} \mu_\infty(\mathcal{O}_\infty H_{r,f}) \asymp \sum_{Nn \leq D(r) < N(n+1)} e^{-\delta D(r) + 2(\delta - \delta_0) \log f \circ D(r)}.$$

Since f is slowly varying, we have

$$\mu_\infty(A_n) \asymp \mathcal{N}''_e(Nn) e^{-\delta Nn + 2(\delta-\delta_0) \log f(Nn)} \asymp f(Nn)^{2(\delta-\delta_0)}.$$

Since f is slowly varying, the sum $\sum_{n \in \mathbb{N}} f(Nn)^{2(\delta-\delta_0)}$ converges if and only if the integral $\int_1^{+\infty} f(t)^{2(\delta-\delta_0)} dt$ converges. This proves Proposition 5.3. □

PROPOSITION 5.5. *There exists a constant $c > 0$ such that if n, m are distinct integers, then*

$$\mu_\infty(A_n \cap A_m) \leq c \mu_\infty(A_n) \mu_\infty(A_m).$$

Proof. We start with the following lemma.

LEMMA 5.6. *For every $A \geq 0$, there exists a constant $c(A) > 0$ such that the following holds. Let X be a CAT(-1) space and ξ_0, ξ_1, ξ_2 be distinct points at infinity of X. Let H_i for $i = 1, 2$ be horospheres centered at ξ_i respectively, bounding disjoint open horoballs.*

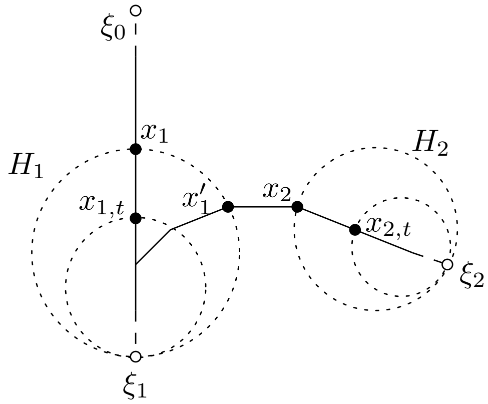

FIGURE 3. Overlapping shadows in trees.

Let x_i be the intersection point with H_i of the geodesic line between ξ_0 and ξ_i. For $t \geq 0$, let $H_{i,t}$ be the horosphere centered at ξ_i, contained in the horoball bounded by H_i and at distance t from H_i. Assume that $\beta_{\xi_0}(x_1, x_2) \leq A$. Let $t \geq c(A)$ be such that $\mathcal{O}_{\xi_0} H_{1,t}$ and $\mathcal{O}_{\xi_0} H_{2,t}$ meet. Then $\mathcal{O}_{\xi_0} H_2$ is contained in $\mathcal{O}_{\xi_0} H_{1,t}$.

Proof. By the techniques of approximation by trees (see [**GH**, p. 33] or [**CDP**, Ch. 8]), this lemma follows from the particular case when X is a tree T (though the constant $c(A)$ might be worse). See Figure 3.

In the case of a tree, one can take $c(A) = A/2 + 1$. Indeed, let $t \geq A/2 + 1$ and for $i = 1, 2$, let $x_{i,t}$ be the intersection point with $H_{i,t}$ of the geodesic line from ξ_0 to ξ_i. Note that x_i is contained in the geodesic ray from ξ_0 to $x_{i,t}$ for $i = 1, 2$. Since $\mathcal{O}_{\xi_0} H_{1,t}$ and $\mathcal{O}_{\xi_0} H_{2,t}$ meet, and by the preliminary remark in the proof of Lemma 5.4, there exists a geodesic line L starting from ξ_0 which passes through both $x_{1,t}$ and $x_{2,t}$.

Suppose that L goes first through $x_{2,t}$, then through $x_{1,t}$. Since H_1, H_2 are disjoint, the points $x_2, x_{2,t}, x_1, x_{1,t}$ are in this order on L and

$$\beta_{\xi_0}(x_1, x_2) = d(x_1, x_2) \geq 2t > A,$$

a contradiction. Hence, L goes first through $x_{1,t}$, then through $x_{2,t}$.

Since H_1 and H_2 are disjoint, the geodesic line L, which enters H_1 at x_1, has to exit H_1 at a point x_1' such that $x_1, x_{1,t}, x_1', x_2, x_{2,t}$ are in this order on L. Hence, every geodesic line starting from ξ_0 which meets H_2 has to go through $H_{1,t}$. This says exactly that $\mathcal{O}_{\xi_0} H_2$ is contained in $\mathcal{O}_{\xi_0} H_{1,t}$. □

Let us now prove Proposition 5.5. With the notation of Lemma 5.6, let $c_3' = c(0)$. Assume that $n < m$. Let R_k be the set of rational lines r with $Nk \leq D(r) < N(k+1)$. To simplify notation, let $\mathcal{O}_{r,f} = \mathcal{O}_\infty H_{r,f}$.

By the reduction argument on f, we may assume that $f(t) \leq e^{-c_3'}$ for every t. By Lemma 5.6, for all rational lines r, r' with $D(r) < D(r')$, if $\mathcal{O}_{r',f}$ meets $\mathcal{O}_{r,f}$, then $\mathcal{O}_\infty H_{r'}$ is contained in $\mathcal{O}_{r,f}$.

Since $A_n = \bigcup_{r\in R_n} \mathcal{O}_{r,f}$, we have

$$\begin{aligned}\mu_\infty(A_m \cap A_n) &\leq \sum_{r\in R_n} \mu_\infty(A_m \cap \mathcal{O}_{r,f}) \\ &\leq \sum_{r\in R_n} \sum_{r'\in R_m : \mathcal{O}_{r',f}\cap\mathcal{O}_{r,f}\neq\emptyset} \mu_\infty(\mathcal{O}_{r',f} \cap \mathcal{O}_{r,f}) \\ &= \sum_{r\in R_n} \sum_{r'\in R_m : \mathcal{O}_{r',f}\cap\mathcal{O}_{r,f}\neq\emptyset} \mu_\infty(\mathcal{O}_{r',f}).\end{aligned}$$

For r in R_n, let I_r be the number of r' in R_m such that $\mathcal{O}_{r',f}$ meets $\mathcal{O}_{r,f}$. By Theorem 4.1 and since f is slowly varying, there exists a constant $c'_4 > 0$ such that $\mu_\infty(\mathcal{O}_{r',f}) \leq c'_4 e^{-\delta Nm+2(\delta-\delta_0)\log f(Nm)}$ for every r' in R_m. Hence,

$$\mu_\infty(A_m \cap A_n) \leq c'_4 e^{-\delta Nm+2(\delta-\delta_0)\log f(Nm)} \sum_{r\in R_n} I_r.$$

The cardinal of R_n, which is $\mathcal{N}''_e(Nn)$, is at most $c'_1 e^{\delta Nn}$. Let us give an upper bound on I_r. By the definition of c'_2 in the proof of Proposition 5.3, for every k in $\mathbb{N}$, the shadows $\mathcal{O}_\infty H_{\rho,c'_2}$ for $\rho \in R_k$ are pairwise disjoint. By Theorem 4.1 and since f is locally bounded, there exists a constant $c'_5 > 0$ such that $\mu_\infty(\mathcal{O}_\infty H_{r',c'_2}) \geq c'_5 e^{-\delta Nm}$ for every r' in R_m. Hence,

$$c'_5 e^{-\delta Nm} I_r \leq \sum_{r'\in R_m : O_{r',f}\cap\mathcal{O}_{r,f}\neq\emptyset} \mu_\infty(\mathcal{O}_\infty H_{r',c'_2}) \leq \mu_\infty(\mathcal{O}_{r,f}),$$

so that $I_r \leq (1/c'_5)e^{\delta Nm}\mu_\infty(\mathcal{O}_{r,f})$. By Theorem 4.1, and since f is slowly varying, there exists a constant $c'_6 > 0$ such that

$$I_r \leq c'_6 e^{\delta Nm} e^{-\delta Nn+2(\delta-\delta_0)\log f(Nn)}.$$

Hence,

$$\begin{aligned}\mu_\infty(A_m \cap A_n) &\leq (c'_4 e^{-\delta Nm+2(\delta-\delta_0)\log f(Nm)})(c'_1 e^{\delta Nn})(c'_6 e^{\delta Nm} e^{-\delta Nn+2(\delta-\delta_0)\log f(Nn)}) \\ &= c'_1 c'_4 c'_6 f(Nn)^{2(\delta-\delta_0)} f(Nm)^{2(\delta-\delta_0)}.\end{aligned}$$

But we have seen in the proof of Proposition 5.3 that $\mu_\infty(A_k) \asymp f(Nk)^{2(\delta-\delta_0)}$. Hence, Proposition 5.5 follows. □

Proof of Theorem 5.1. For every rational line r and every geodesic line ξ starting from ∞ in M_0, let $d'_\infty(\xi, r)$ be the lower bound of the e^{-t} for $t > 0$ such that ξ meets $H_{r,t}$. That is $d'_\infty(\xi, r) \leq e^{-t}$ if and only if ξ meets $H_{r,t}$.

According to [**HP2**], there is a constant $c'_7 > 0$ such that

$$\frac{1}{c'_7} d_\infty(\xi, r) \leq e^{-D(r)} d'_\infty(\xi, r) \leq c'_7 d_\infty(\xi, r).$$

In particular, if the endpoint of ξ belongs to $\mathcal{O}_\infty H_{r,f}$, then $d'_\infty(\xi, r) \leq e^{-(-\log f\circ D(r))}$, hence $d_\infty(\xi, r) \leq c'_7 f \circ D(r)e^{-D(r)}$. Conversely, if $d_\infty(\xi, r) \leq (1/c'_7) f \circ D(r)e^{-D(r)}$, then $d'_\infty(\xi, r) \leq e^{-(-\log f\circ D(r))}$; hence, the endpoint of ξ belongs to $\mathcal{O}_\infty H_{r,f}$.

Define $A_\infty = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$, which is the set of points in ∂M_0 belonging to infinitely many A_n's. Note that A_∞ is contained in the subset $\Gamma_0\backslash(\Lambda\Gamma - \{\xi_0\})$, since the orbit under Γ of the parabolic point ξ_0 is dense in the limit set of Γ.

By these arguments, if the endpoint of ξ belongs to A_∞, then there are infinitely many rational lines r such that $d_\infty(\xi, r) \leq c'_7 f \circ D(r)e^{-D(r)}$. And if there are infinitely many rational lines r such that $d_\infty(\xi, r) \leq (1/c'_7) f \circ D(r)e^{-D(r)}$, then ξ belongs to A_∞. With the notation in the statement of Theorem 5.1, we then have

$$E\left(\frac{1}{c'_7} f\right) \subset A_\infty \subset E(c'_7 f).$$

Note that the convergence or divergence of the integral $\int_1^\infty f^{2(\delta-\delta_0)}$ is unchanged if one replaces f by λf for any $\lambda > 0$. Hence, to prove that $\mu_\infty(E(f)) > 0$ if and only if $\int_1^\infty f^{2(\delta-\delta_0)}$ diverges, it is sufficient to prove that $\mu_\infty(A_\infty) > 0$ if and only if $\int_1^\infty f^{2(\delta-\delta_0)}$ diverges.

We use the following result whose proof can be found, for instance, in [**Spr**].

THEOREM 5.7. *Let (Y, ν) be a probability space. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of Y such that there exists a constant $c > 0$ with $\nu(B_n \cap B_m) \leq c\nu(B_n)\nu(B_m)$ for all distinct integers n, m. Let $B_\infty = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} B_k$. Then $\nu(B_\infty) > 0$ if and only if $\sum_{n=0}^\infty \nu(B_n)$ diverges.*

We now use Propositions 5.5 and Proposition 5.3 and apply the previous result with $Y = \partial M_0$, $\nu = \mu_\infty$, $B_n = A_n$, to obtain that $\mu_\infty(E(f)) > 0$ if and only if $\int_1^\infty f^{2(\delta-\delta_0)}$ diverges. This is the first conclusion of Theorem 5.1.

Assume that $\int_1^\infty f^{2(\delta-\delta_0)}$ diverges. Let us prove that $\mu_\infty({}^c E(f)) = 0$, which proves the second conclusion of Theorem 5.1.

Let $g : [0, +\infty[\to]0, +\infty[$ be a map decreasing to 0 such that $\int_1^\infty (gf)^{2(\delta-\delta_0)}$ diverges. Let $E'(f)$ be the set of geodesic lines ξ in M_0 starting from ∞ such that there exist $c > 0$ and infinitely many rational lines r in M_0 with $d_\infty(\xi, r) \leq cf(D(r))e^{-D(r)}$. Since $E(gf) \subset E'(gf)$, the first conclusion of Theorem 5.1 implies that $\mu_\infty(E'(gf)) > 0$. It is clear that the union of $\{\xi_0\}$ and of the pre-image in $\partial\widetilde{M}$ of $E'(gf) \subset \partial M_0$ is invariant under Γ.

Since Γ is of divergent type, the action of Γ on $\partial\widetilde{M}$ for the Patterson–Sullivan measure is ergodic, see, for instance, [**Rob**]. By [**DOP**], since ξ_0 is a bounded parabolic point and Γ is non-elementary of divergent type, the measure μ_∞ has no atom at ξ_0. By ergodicity, $\mu_\infty({}^c E'(gf)) = 0$. But $E'(gf) \subset E(f)$ since g is decreasing to 0. Hence $\mu_\infty({}^c E(f)) = 0$. □

6. *The logarithm law for the geodesic flow in variable curvature*

Define a map $\Delta_e : M \to \mathbb{R}$ that describes the penetration distance into the maximal Margulis neighbourhood V_e of the cusp e, by $\Delta_e(x) = -1$ if x does not belong to V_e, and $\Delta_e(x) = d(x, \partial V_e)$ otherwise.

COROLLARY 6.1. *With the notation of §2, assume that e is a bounded cusp,* $f_{\widetilde{\pi}}(t) \asymp e^{\delta t}$*, and* $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$*. For every y in M and almost every v in* $T^1_y(M)$ *(for the Patterson–Sullivan measure), we have*

$$\limsup_{t\to+\infty} \frac{\Delta_e(\gamma_v(t))}{\log t} = \frac{1}{2(\delta-\delta_0)}.$$

Proof. We will apply Theorem 5.1 to the functions $f_\kappa(t) = t^{-\kappa}$. Note that the integral $\int_1^\infty (f_\kappa)^{2(\delta-\delta_0)}$ diverges if and only if $\kappa \leq 1/2(\delta-\delta_0)$.

In what follows, the variable ξ denotes a geodesic line starting from ∞ in M_0, with endpoint in the (full measure for μ_∞) image in ∂M_0 of $\Lambda\Gamma - \{\xi_0\}$. Take as the origin $\xi(0)$ on ξ its intersection with H_∞.

By the definition of the Hamenstädt distance, there exists a constant $c''_1 \geq 0$ such that, for every ξ and every rational line r in M_0 such that ξ enters HB_r, if ξ_r is the tangency point to some $H_{r,t}$ for some $t = t_{\xi,r}$ (i.e. ξ_r is the deepest penetration point of ξ in HB_r), then

$$e^{-D(r)-t-c''_1} \leq d_\infty(\xi, r) \leq e^{-D(r)-t+c''_1}. \qquad (\sharp)$$

With this notation, there also exists a constant c''_2 such that the length $\ell_\xi(r)$ of the subsegment between the origin of ξ and ξ_r satisfies

$$D(r) + t_{\xi,r} - c''_2 \leq \ell_\xi(r) \leq D(r) + t_{\xi,r} + c''_2.$$

If $\pi_0 : M_0 \to M$ is the canonical covering map, by the properties of the maximal Margulis neighbourhood, and since e is bounded, there exists a constant $c''_3 \geq 0$ such that

$$t_{\xi,r} - c''_3 \leq \Delta_e(\pi_0(\xi_r)) \leq t_{\xi,r} + c''_3.$$

Let $\kappa_n = 1/2(\delta-\delta_0) + 1/n$. By the first part of Theorem 5.1, for almost every ξ, except for finitely many rational lines r, $d_\infty(\xi, r) \geq f_{\kappa_n}(D(r))e^{-D(r)}$. Hence, by the formula ($\sharp$), for almost every ξ, we have $t_{\xi,r} \leq \kappa_n \log D(r) + c''_1$ for every (except finitely many) rational line r such that ξ enters HB_r. In particular, for almost every ξ for every real constant c, as r goes to infinity in the discrete set of rational lines with ξ meeting HB_r, we have $\log D(r) \sim \log(D(r) + t_{\xi,r} + c)$. Therefore, for almost every ξ,

$$\limsup \frac{\Delta_e(\pi_0(\xi_r))}{\log \ell_\xi(r)} \leq \kappa_n,$$

where the upper bound is taken as r goes to infinity in the discrete set of rational lines with ξ meeting HB_r.

Similarly, by the second part of Theorem 5.1 using the function f_κ with $\kappa = 1/2(\delta-\delta_0)$, for almost every ξ,

$$\limsup \frac{\Delta_e(\pi_0(\xi_r))}{\log \ell_\xi(r)} \geq \kappa,$$

where the upper bound is taken as before. Removing countably many sets of measure zero, we get that, for almost every ξ,

$$\limsup \frac{\Delta_e(\pi_0(\xi_r))}{\log \ell_\xi(r)} = \kappa. \qquad (\dagger)$$

Now let y be a point in M, and choose a lift $\widetilde{y}$ of y in $\widetilde{M}$. Let $v \mapsto \widetilde{v}$ be the map $T^1_y M \to T^1_{\widetilde{y}} \widetilde{M}$ induced by the covering map $\widetilde{\pi}$. Note that ξ_0 is not an atom for the Patterson–Sullivan measures (see [**DOP**]).

The complement in $T^1_y M$ of the vector v_0 such that $\widetilde{v}_0$ points towards ξ_0, which has full Patterson–Sullivan measure, can be covered by countably many open subsets U such that the following holds: there exists a relatively compact small open subset $\widetilde{U}$ of geodesic lines in $\widetilde{M}$ starting from ξ_0, with $\widetilde{U}$ embedding in M_0 under $\widetilde{\pi}_0$, such that U is the subset of vectors v in $T^1_y M$ such that $\tilde{v}$ points towards the endpoint of some element $\widetilde{\zeta} = \widetilde{\zeta}(v)$ in $\widetilde{U}$.

Note that $\gamma_{\widetilde{v}}$ and $\widetilde{\zeta}$ become arbitrarily close towards their common point at infinity, so that the geodesic ray and line γ_v and $\widetilde{\pi}(\widetilde{\zeta})$ have the same asymptotic behaviour inside the maximal Margulis neighbourhood V_e of e.

Let $\zeta = \zeta(v)$ be the image of $\widetilde{\zeta}$ in M_0. Since $\mu_{\widetilde{y}}$ and μ_{ξ_0} are absolutely continuous, the map $v \mapsto \zeta$ (which is a homeomorphism onto its image) preserves the sets of measure zero. For every v in U, if $t \geq 0$ is such that $\gamma_v(t)$ is the maximal penetration point in V_e of some connected component of $\mathrm{int}(V_e) \cap \gamma_v(\mathbb{R})$, then there is a constant $c''_4 \geq 0$ and a rational line r such that

$$|\Delta_e(\pi_0(\zeta_r)) - \Delta_e(\gamma_v(t))| \leq c''_4.$$

Note that there is a constant $c''_5 \geq 0$ such that

$$|t - \ell_\zeta(r)| \leq |d_{\widetilde{M}}(\widetilde{y}, \gamma_{\widetilde{v}}(t)) - d(\gamma_{\widetilde{v}}(t), \widetilde{\zeta}(0))| + c''_5 \leq d_{\widetilde{M}}(\widetilde{y}, \widetilde{\zeta}(0)) + c''_5,$$

which is uniformly bounded.

Hence Corollary 6.1 follows from formula (†). □

COROLLARY 6.2. *With the notation of §2, assume that M is geometrically finite, and that $f_{\widetilde{\pi}_0}(t) \asymp e^{\delta_0 t}$ and similarly for every cusp. Assume that δ_0 is the biggest critical exponent of the parabolic subgroups of the cusps of M. For every y in M and almost every v in $T^1_y M$ (for the Patterson–Sullivan measure),*

$$\limsup_{t \to +\infty} \frac{d_M(y, \gamma_v(t))}{\log t} = \frac{1}{2(\delta - \delta_0)}.$$

Theorem 1.4 in the introduction immediately follows from this corollary.

Proof. Note that $f_{\widetilde{\pi}}(t) \asymp e^{\delta t}$ by Lemma 3.6. Since M is geometrically finite, it has only finitely many cusps and $\widetilde{\pi}(C\Lambda\Gamma)$ is the union of a compact subset and the (intersections with $\widetilde{\pi}(C\Lambda\Gamma)$ of the) finitely many maximal Margulis neighbourhoods of the cusps. The result then follows from Corollary 6.1, by considering the excursions of the geodesics in the different cusps. □

Acknowledgements. We thank T. Nagnibeda and S. Grigorchuk for suggesting Theorem 3.1 to the second author. We thank the I.H.E.S. for its support and hospitality during a stay by the first author in 1999–2000. We thank the referee for many very helpful comments. Research by Sa'ar Hersonsky is supported in part by a grant from The David and Elaine Potter Charitable Foundation.

REFERENCES

[BHP] K. Belabas, S. Hersonsky and F. Paulin. Counting horoballs and rational geodesics. *Bull. Lond. Math. Soc.* **33** (2001), 606–612.

[Bou] M. Bourdon. Structure conforme au bord et flot géodésique d'un CAT(-1) espace. *L'Ens. Math.* **41** (1995), 63–102.

[Bow] B. Bowditch. Geometrical finiteness with variable negative curvature. *Duke Math. J.* **77** (1995), 229–274.

[BuK] P. Buser and H. Karcher. *Gromov's Almost Flat Manifolds (Astérisque, 81)*. Société Mathématique de France, 1981.

[CDP] M. Coornaert, T. Delzant and A. Papadopoulos. *Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov (Lecture Notes in Mathematics, 1441)*. Springer, Berlin, 1990.

[Dal] F. Dal'Bo. Remarques sur le spectre des longueurs d'une surface et comptage. *Bol. Soc. Bras. Math.* **30** (1999), 199–221.

[DOP] F. Dal'Bo, J.-P. Otal and M. Peigné. Séries de Poincaré des groupes géométriquement fini. *Israel J. Math.* **118** (2000), 109–124.

[Fuk] K. Fukaya. Hausdorff convergence of Riemannian manifolds and applications. *Recent Topics in Differential and Analytic Geometry (Advanced Studies in Pure Mathematics, 18)*. Ed. T. Ochiai. Kinokuniya Company, 1990, pp. 143–238.

[GH] E. Ghys and P. de la Harpe (Eds.). *Sur les groupes hyperboliques d'après Mikhael Gromov (Progress in Mathematics, 83)*. Birkhäuser, 1990.

[GK] R. I. Grigorchuk and P. F. Kurchanov. Some questions of group theory related to geometry. *Algebra VII (Encyclopedia of Mathematical Science, 58)*. Eds. A. N. Parshin and I. R. Shafarevich. Springer, Berlin, 1993, pp. 167–232.

[Gro1] M. Gromov. Groups of polynomial growth and expanding maps. *Publ. Math. Inst. Hautes Études Sci.* **53** (1981), 53–78.

[Gro2] M. Gromov, J. Lafontaine and P. Pansu. *Metric Structures for Riemannian Manifolds*. Birkhaüser, 2000.

[Ham] U. Hamenstädt. A new description of the Bowen–Margulis measure. *Ergod. Th. & Dynam. Sys.* **9** (1989), 455–464.

[HP1] S. Hersonsky and F. Paulin. On the rigidity of discrete isometry groups of negatively curved spaces. *Comm. Math. Helv.* **72** (1997), 349–388.

[HP2] S. Hersonsky and F. Paulin. Diophantine approximation for negatively curved manifolds. *Math. Z.* **241** (2002), 181–226.

[HP3] S. Hersonsky and F. Paulin. Diophantine approximation on negatively curved manifolds and in the Heisenberg group. *Rigidity in Dynamics and Geometry (Cambridge, 2000)*. Eds. M. Burger and A. Iozzi. Springer, Berlin, 2002, pp. 203–226.

[HV] R. Hill and S. Velani. The Jarnìk–Besicovitch theorem for geometrically finite Kleinian groups. *Proc. London Math. Soc.* **77** (1998), 524–550.

[Kh] A. Khinchine. *Continued Fractions*. University of Chicago Press, Chicago, 1964.

[KM] D. Kleinbock and G. Margulis. Logarithm laws for flows on homogeneous spaces. *Invent. Math.* **138** (1999), 451–494.

[Man] A. Manning. Topological entropy for geodesic flows. *Ann. Math.* **110** (1979), 567–573.

[Mar] G. Margulis. Applications of ergodic theory for the investigation of manifolds of negative curvature. *Funct. Anal. Appl.* **3** (1969), 335–336.

[Pat] S. J. Patterson. The limit set of a Fuchsian group. *Acta. Math.* **136** (1976), 241–273.

[Rob] T. Roblin. *Ergodicité du Feuilletage Horocyclique, Mélange du flot Géodésique et Équidistribution pour les Groupes Discrets en Courbure Négative (Mémoires du Société Mathématiques de France)*. To appear.

[Spr] V. G. Sprindzǔk. *Mahler's Problem in Metric Number Theory (Translations of Mathematical Monographs, 25)*. American Mathematical Society, Providence, 1969.

[SV] B. Stratmann and S. L. Velani. The Patterson measure for geometrically finite groups with parabolic elements, new and old. *Proc. London Math. Soc.* **71** (1995), 197–220.

[Sul1] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 172–202.

[Sul2] D. Sullivan. Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics. *Acta Math.* **149** (1982), 215–237.

[Sul3] D. Sullivan. Entropy, Hausdorff measures old and new, and the limit set of geometrically finite Kleinian groups. *Acta. Math.* **153** (1983), 259–277.