Comment. Math. Helv. 72 (1997) 349–388
0010-2571/97/030349-40 \$ 1.50+0.20/0

Commentarii Mathematici Helvetici

On the rigidity of discrete isometry groups of negatively curved spaces

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Abstract. We prove an ergodic rigidity theorem for discrete isometry groups of CAT(-1) spaces. We give explicit examples of divergence isometry groups with infinite covolume in the case of trees, piecewise hyperbolic 2-polyhedra, hyperbolic Bruhat-Tits buildings and rank one symmetric spaces. We prove that two negatively curved Riemannian metrics, with conical singularities of angles at least 2π, on a closed surface, with boundary map absolutely continuous with respect to the Patterson-Sullivan measures, are isometric. For that, we generalize J.-P. Otal's result to prove that a negatively curved Riemannian metric, with conical singularities of angles at least 2π, on a closed surface, is determined, up to isometry, by its marked length spectrum.

Mathematics Subject Classification (1991). 57S30, 53C45, 20H10, 51M10, 51E24.

Keywords. Negative curvature, divergence group, Patterson–Sullivan measure, marked length spectrum, hyperbolic building, discrete group, rigidity, surface with conical singularities.

Introduction

Mostow's global rigidity theorem [Mos] asserts that two closed locally symmetric manifolds having isomorphic fundamental group are isometric, except for real hyperbolic surfaces. In the case of real hyperbolic (i.e. of constant curvature -1) manifolds (of dimension $n \geq 3$), where both manifolds are covered by the hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$, the first step is to construct an equivariant boundary map $\tilde{\phi} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ and to prove that $\tilde{\phi}$ is sufficiently regular (here quasi-conformal). In particular, under the hypothesis on the dimension, the boundary map is absolutely continuous with respect to the Lebesgue measure. Then by using dynamical-ergodic properties of the action of the groups at infinity, Mostow proved that the map is conformal, i.e. belongs to the Möbius group, hence is the extension of an equivariant isometry.

Mostow's rigidity theorem has been extended by [Pra] to the noncompact finite volume case. In his seminal work (see [Sul1,Sul2]) on the ergodic theory at infinity of discrete isometry groups of $\mathbb{H}^n_{\mathbb{R}}$, D. Sullivan [Sul2] extended Mostow's rigidity result to real hyperbolic n-manifolds whose volume grows slower than the volume of $\mathbb{H}^n_{\mathbb{R}}$. P. Tukia [Tuk] further extended it to the quotients by divergence groups of isometries of $\mathbb{H}^n_{\mathbb{R}}$ (see the definition below).

The first goal of this paper is to generalize Mostow's result to divergence groups of isometries of CAT(-1) metric spaces X (see [Gro1, GH] for a definition). These spaces include simply connected Riemannian manifolds with sectional curvature ≤ -1 (for instance negatively curved symmetric spaces), simplicial trees and some simply connected piecewise hyperbolic polyhedra (see section 2). They can be compactified by adding a *boundary* (or space at infinity) ∂X (see section 1).

Recall that (see for instance [Bou1,Bou2]) if Γ is a discrete group of isometries of X, then its *Poincaré series* is, with $s \in \mathbb{R}$ and $x \in X$,

$$P(s,x) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)}.$$

This series converges if $s > \delta$ and diverges if $s < \delta$, for some δ (independent of x) called the *critical exponent* of Γ. At $s = \delta$, the Poincaré series may converge or diverge. In the latter case, if δ is finite and non zero, the group Γ is called a *divergence group.*

Let Γ be a divergence group. Consider the measure $\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} D_{\gamma x}$, where D_y is the Dirac mass at the point $y \in X$, normalized to be a probability measure. It converges weakly as $s \to \delta^+$ to a measure μ_x, whose support is the *limit set* of Γ, contained in ∂X. This measure is called the Patterson-Sullivan measure (see [Pat1,Sul1] in the real hyperbolic case, and [Bou1,Bou2,BuMo] in the case of CAT(-1) spaces).

One may associate to four distinct points on the space at infinity of X, a non-negative number called the *crossratio* (see [Ota2] in the negatively curved manifold case, [Pau1] for general Gromov hyperbolic spaces, and section 1). In the case of $X = \mathbb{H}^3_{\mathbb{R}}$, it is the absolute value of the classical complex crossratio. A map is called *Möbius* if it preserves the crossratios. Our first main result is:

Theorem A. *Let X_1, X_2 be locally compact complete CAT(-1) metric spaces. Let Γ_1 and Γ_2 be discrete groups of isometries of X_1, X_2, having the same critical exponent. Suppose that Γ_2 is a divergence group. Let $\tilde\phi : \partial X_1 \to \partial X_2$ be a Borel map, equivariant for some morphism $\Gamma_1 \to \Gamma_2$, which is non-singular with respect to the Patterson-Sullivan measures.*

Then $\tilde\phi$ is Möbius on the limit set of Γ_1.

Note that the hypothesis on the critical exponents is necessary in order to prevent X_2 being X_1 with a scaled metric. Specializing X_1 and using work of M. Bourdon [Bou3,Bou4], we obtain:

Corollary B. *Under the above hypotheses, assume moreover that one of the following situations holds:*

1. *X_1 is a rank one symmetric space, of curvature normalized to have maximum -1, and the limit set of Γ_1 equals ∂X_1.*

2. X_1 *is a Bruhat-Tits building modeled on a discrete reflection group of* $\mathbb{H}^n_{\mathbb{R}}$ *(see section 2), and the limit set of* Γ_1 *equals* ∂X_1.

Then $\tilde{\phi}$ *is the extension of an isometric embedding of* X_1 *into* X_2.

In the case when X_1, X_2 are both locally finite trees, these results are due to M. Coornaert [Coo2]. While we were writing this paper, we received a preprint of C. Yue [Yue], which proves in the case X_1, X_2 are both rank one symmetric spaces a stronger theorem than Theorem A: let Γ_1 and Γ_2 be discrete groups of isometries of X_1, X_2, with Γ_2 a divergence group; let $\tilde{\phi} : \partial X_1 \to \partial X_2$ be a Borel map, equivariant for some morphism $\Gamma_1 \to \Gamma_2$, which is non-singular with respect to the Patterson-Sullivan measures; then $\tilde{\phi}$ is the extension of an equivariant isometric embedding of X_1 into X_2. Because of the special geometry on the boundary, C. Yue does not need to require a priori that the critical exponents are equal. But as our examples show (see section 2), trees and rank one symmetric spaces are not the only applications.

Recall that in a locally compact CAT(-1) space M, the map which associates to a free homotopy class of loops, the length of the unique closed geodesic contained in it, is called the *marked length spectrum* of M. The second goal of this paper is to prove the following result.

Theorem C. *Let* S_1, S_2 *be two closed connected surfaces, having negatively curved Riemannian metrics with finitely many conical singularities of angle at least* 2π. *Let* $\phi : S_1 \to S_2$ *be an homeomorphism. The following assertions are equivalent:*

1. *the boundary map is absolutely continuous with respect to the Patterson-Sullivan measures, and* S_1, S_2 *have the same volume entropy*
2. *the boundary map is Möbius,*
3. S_1, S_2 *have marked length spectrum in one-to-one correspondance via* ϕ,
4. S_1 *and* S_2 *are isometric, by an isometry homotopic to* ϕ.

In the non-singular case, the equivalence between (1) and (4) is due to T. Kuusalo [Kuu] in constant curvature, and the equivalence between (3) and (4) is due to J.-P. Otal [Ota1]. See also [Lal] for partial results.

The paper is organized as follows. In section 1, we review well known material on CAT(-1) spaces, their boundaries, crossratios, Patterson-Sullivan measures and divergence groups. In section 2, we give new examples of divergence groups for the complex hyperbolic plane, for simplicial homogeneous trees, for piecewise hyperbolic polygonal 2-complexes with uncountable automorphism groups, and for hyperbolic Bruhat-Tits buildings. In section 3, we prove Theorem A, following Sullivan's method, and Corollary B.

Finally, in section 4, we give the proof of Theorem C. One of the main new ingredients is the new notion of *Möbius (geodesic) currents* at infinity for CAT(-1) surfaces (see subsection 4.1). We refer to [Bon1,Bon2] for definitions of geodesic currents on the boundary of CAT(-1) spaces. We then have to prove "measurable" extensions of J.-P. Otal's arguments [Ota1], in particular concerning the Liouville current. We conclude as in [Ota1]. Note that Theorem C also holds for other singular negatively curved Riemannian metrics with geodesic flow having singularities on a measure 0 subset, in a sense we will not make precise.

In the appendix, we introduce on the boundary minus a point, a natural metric, analogous to the euclidean metric in the real hyperbolic case, or to the Cygan metric for others rank one symmetric spaces. From this, we derive some relations between crossratios and the marked length spectrum. We prove that the Möbius maps fixing the point are homotheties for these metrics.

1. Review of CAT(-1) spaces

For the content of this section, only recalled here to make the reading of the paper easier, the reader is refered to [GH] and [Bou1,Bou2,BuMo].

Let X be a CAT(-1) space which is *proper* (i.e. with compact closed balls). This is equivalent to requiring X to be complete and locally compact.

Define an equivalence relation on the set of geodesic rays in X: two geodesic rays are *asymptotic* if their Hausdorff distance is finite. The set of equivalence classes is called the *boundary* of X and will be denoted by ∂X. In fact, if r, r' are two asymptotic rays, then the distance from $r(t)$ to (the image of) r' goes to zero as t goes to infinity.

Note that between any two points in $X \cup \partial X$ there exists a unique geodesic.

There is a natural topology on $X \cup \partial X$, which turns it into a compact metrizable space, in which X is open and dense. The idea is that two geodesic rays are close if they are at distance less than a given constant for a long time. In particular, any isometry extends continuously to a homeomorphism of $X \cup \partial X$, denoted by the same letter.

The isometries g of X are classified into three types, *elliptic*, *parabolic*, *hyperbolic*. The *translation length* of g is $\ell(g) = \inf_{x \in X} d(x, gx)$. If g is an hyperbolic isometry of X, then g is an homeomorphism of ∂X with a North-South dynamics: g has exactly two fixed point g_-, g_+ in ∂X such that for any neighborhood U_-, U_+ of respectively g_-, g_+, there exists a positive power of g mapping the complement of U_- into U_+. The *translation axis* is the geodesic between g_- and g_+, and is precisely the set of points x such that $d(x, gx) = \ell(g)$.

Let $x, y \in X$, and $a \in \partial X$. The *horospherical distance* (or *Buseman function*) between x and y with respect to a is defined by

$$B_a(x,y) = \lim_{t \to \infty} \left(d(x, r(t)) - d(y, r(t)) \right). \qquad (1)$$

The limit exists and does not depend on the choice of a geodesic ray r ending at a. The Buseman functions satisfy the following cocycle relation

$$B_a(x,y) + B_a(y,z) = B_a(x,z).$$

The *horosphere* centered at $a \in \partial X$, passing through $x \in X$, is the set of $y \in X$ such that $B_a(x, y) = 0$. A geodesic having a as one endpoint, cuts each horosphere centered at a in one and only one point. An isometry γ maps horospheres centered at a to horospheres centered at γa.

Let $a, b \in \partial X$ and let $x \in X$. Define the *Gromov product* of a, b with respect to x by:

$$\langle a, b\rangle_x = \lim_{a_i \to a, b_i \to b} \frac{1}{2}\left(d(a_i, x) + d(x, b_i) - d(a_i, b_i)\right). \tag{2}$$

The limit exists and does not depend on the choice of sequences of points a_i, b_i in X tending to a, b. As in [Kai1], $\langle a, b\rangle_x = \frac{1}{2}\left(B_a(x, p) + B_b(x, p)\right)$ for any point p on the geodesic between a and b.

For $x \in X$ and $a, b \in \partial X$, we define the *visual distance* between a and b viewed from x as

$$d_x(a, b) = \begin{cases} e^{-\langle a, b\rangle_x} & \text{if } a \neq b \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The family $\{d_x\}_{x \in X}$ is a *conformal family of metrics*, i.e. it satisfies the following properties (see [Bou1,Bou2]).

1. ([Bou1, Theorem 2.5.1]) For every $x \in X$, d_x is a distance on ∂X, and for any isometry g of X, we have $d_{gx}(ga, gb) = d_x(a, b)$,
2. ([Bou1, Corollary 2.6.3]) For $x, y \in X$, the metrics d_x, d_y are in the same conformal class, that is more precisely, for $a \neq b$:

$$\frac{d_y(a, b)}{d_x(a, b)} = e^{\frac{1}{2}(B_a(x,y) + B_b(x,y))}. \tag{4}$$

In particular, for any isometry g of X and for any $x \in X$, g acts conformally on $(\partial X, d_x)$. The conformal factor at a point $a \in \partial X$ is (the limit does exist):

$$\lim_{b \to a} \frac{d_x(gb, ga)}{d_x(b, a)} = e^{B_a(x, g^{-1}x)}$$

and will be denoted by $j_x g(a)$.

Therefore, we immediately obtain the following *mean value formula*, showing the exact amount by which the visual metric is distorted by an isometry. Let g be an isometry of X and $a, b \in \partial X$, then

$$d_x(ga, gb)^2 = j_x g(a) j_x g(b) d_x(a, b)^2. \tag{5}$$

In [Pau1,Bou3], generalizing work of [Ota2], the *crossratio* of four distinct points a, b, c, d on the boundary of X is defined by

$$[a, b, c, d] = \lim_{a_i \to a, b_i \to b, c_i \to c, d_i \to d} \frac{1}{2}\left(d(a_i, c_i) - d(c_i, b_i) + d(b_i, d_i) - d(d_i, a_i)\right) \tag{6}$$

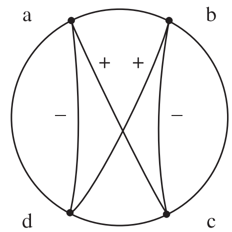

Figure 1. Crossratio of four points on the boundary

The limit exists and does not depend on the choice of sequences a_i, b_i, c_i, d_i in X tending to a, b, c, d respectively. Note that any isometry of X preserves the crossratio.

The following formula holds (and is in particular independant of x in X):

$$e^{[a,b,c,d]} = \frac{d_x(a,c)d_x(b,d)}{d_x(b,c)d_x(a,d)}\,. \tag{7}$$

This implies that the crossratios have the following symmetries:

$$[a,b,c,d] = [c,d,a,b], \quad [a,b,c,d] = -[b,a,c,d], \quad [a,b,c,d] = -[a,b,d,c].$$

Definition 1.1. *Let X_1, X_2 be CAT(-1) spaces. Let A be a subset of ∂X_1. An injective map $f : A \to \partial X_2$ is called Möbius if it preserves the crossratios of quadruples of distinct points of A.*

Let Γ be a discrete subgroup of isometries of X. The *limit set* of Γ is the subset of ∂X of accumulation points of any orbit of Γ in X.

Generalizing Patterson's construction for Fuchsian groups [Pat1] and Sullivan's [Sul1,Sul2] for discrete subgroups of isometries of the real hyperbolic n-space, there exists a remarkable family of measures on ∂X associated to Γ, see [Bou2,BuMo].

A *conformal density* of dimension $\delta \in \mathbb{R}^+$ for Γ is a family $\{\mu_x\}_{x\in X}$ of pairwise absolutely continuous mesures, whose support is the limit set of Γ, satisfying the following properties.

- $g_*\mu_x = \mu_{gx}$ for all g in Γ and $x \in X$,
- For all $a \in \partial X$, $x, y \in X$,

$$\frac{d\mu_x}{d\mu_y}(a) = e^{-\delta B_a(x,y)} \tag{8}$$

The measure μ_x is called the *Patterson-Sullivan measure* measure on ∂X viewed from x. If g is an isometry, then $g_*\mu_x = j_x(g^{-1})(a)^\delta \mu_x$.

Let $\delta = \delta(\Gamma)$ be the critical exponent of the Poincaré series of Γ (as defined in the introduction). If $\delta \neq 0, \infty$, then such a conformal density exists, whether the Poincaré series diverges or not. See the above references, as well as [Coo1] for a quasi-extension to word hyperbolic groups. Note that under a scaling of the metric, if d is replaced by λd with $\lambda > 0$, then the critical exponent δ becomes $\frac{1}{\lambda}\delta$, but the Patterson-Sullivan measure at any point x is unchanged.

If X admits an action by a discrete cocompact group of isometries, then any discrete group of isometries (whose limit set contains at least 3 points) has a critical exponent different from $0, \infty$.

The diagonal action of Γ on $\partial X \times \partial X - \Delta$ has an invariant measure ν, called the *Bowen-Margulis measure* (see [Bou1,Bou2,Kai1,Kai2] and [BuMo, section 6]), analogous to Sullivan's construction in the real hyperbolic case:

$$d\nu(a,b) = \frac{d\mu_x(a)\, d\mu_x(b)}{d_x(a,b)^{2\delta}}. \tag{9}$$

This measure does not depend on x by (8) and (4), and in particular is invariant by the diagonal action of Γ.

2. Examples of non geometrically finite divergence groups

Classical examples of divergence groups Γ of CAT(-1) spaces are (see for instance [Bou1,Bou2]) the discrete groups of isometries that are *convex cocompact*, i.e. such that the quotient of the convex hull of the limit set by the group is compact. This is in particular the case if Γ is cocompact.

In this section, we give examples of divergence groups that are not convex cocompact, for rank one symmetric spaces, locally finite trees, negatively curved polygonal 2-complexes and hyperbolic Bruhat-Tits buildings.

2.1. Negatively curved symmetric spaces

The criteria we are using is the following one :

Theorem 2.1. *Let Γ be a discrete torsion-free group of isometries of a negatively curved symmetric space X. The following conditions are equivalent:*
1. *the Brownian motion on $M = X/\Gamma$ is recurrent*
2. *M has no (finite) Green function*
3. *Γ is a divergence group and the critical exponent of Γ is the volume entropy of X*

$$h_X = \limsup_{k\to\infty} \frac{1}{k} \log Vol\ B_X(x,k)$$

where $B_X(x,k)$ is the ball of radius k and center any x in X.

Proof. Let $G_M(x,y)$ be the Green function of M and $p_t(x,y)$ be the heat kernel on M. By definition,

$$G_M(x,y) = \int_0^{+\infty} p_t(x,y)dt$$

is the expected probability for the Brownian motion starting at x to end at y. Thus the equivalence between (1) and (2) is clear.

To prove the equivalence of (2) and (3), recall that, by invariance under isometries, for a rank one symmetric space X, the Green function $G_X(x,y)$ of X depends only on the distance between x and y. Apply the Green formula

$$-f(x) = \int_\Omega G(x,y)\Delta f(y)dv(y) + \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y)f(y)ds(y)$$

where f is the constant function 1 and Ω is the ball of center x and radius R. So if $G_X(x,y) = \phi(d(x,y))$, one has

$$\phi'(R) = -\frac{1}{\text{Vol } S_X(x,R)}.$$

Hence one gets the following inequality, with $h = h_X$:

$$Ae^{-hd(x,y)} \leq G_X(x,y) \leq Be^{-hd(x,y)}$$

for some positive constants A, B, and for $d(x,y) \geq 1$ (see also [Led]). Now, the Green function $G_M(x,y)$ of M is obviously

$$G_M(x,y) = \sum_{\gamma\in\Gamma} G_X(x,\gamma y).$$

So the result follows from the well known fact that the critical exponent of a discrete subgroup of isometries of X is at most h. $\square$

Corollary 2.2. *Let Γ be a discrete cocompact group of isometries of a rank one symmetric space X, and let Γ_1 be the kernel of a morphism of Γ onto H. Then Γ_1 is a divergence group with critical exponent h_X if and only if H is virtually $0, \mathbb{Z}, \mathbb{Z}^2$.*

Note that by a very long argument, M. Rees ([Ree]) proved a special case of this corollary, that if Γ_1 is a normal subgroup of a discrete cocompact group Γ of isometries of $\mathbb{H}^n_{\mathbb{R}}$, such that $\Gamma/\Gamma_1 \cong \mathbb{Z}^\nu$ where $\nu \leq 2$, then Γ_1 is of divergence type. Particular cases were also proven in [LySu, Theorem 4] and [Gui].

Proof. The Riemannian manifold $M_1 = X/\Gamma_1$ has *bounded geometry*, i.e. its curvature is bounded from below and its injectivity radius is positive (this follows from the fact that Γ_1 is a subgroup of a cocompact group, hence cannot have arbitrarily short geodesics). It is proved in [Kan] that M_1 has a recurrent Brownian motion if and only if the simple random walk on any $(1,3)$-net in M_1 is recurrent. An $(1,3)$-*net* in M is any graph whose set of vertices is a maximal subset of M whose points are pairwise at distance at least 1, and with a vertex between two vertices at distance at most 3. But since Γ is cocompact, it is quasi-isometric to X, hence M_1 is quasi-isometric to Γ/Γ_1, that is to H. Note that by [Kan], the simple random walks on two locally finite graphs (with uniformely bounded degrees) that are quasi-isometric are simultaneously recurrent or non recurrent. (See also [VSC, Theorem X.3.1 page 142] stating that the Brownian motion on M_1 is transient if and only if the covering group Γ/Γ_1 is transient). The result follows then from A. Varopoulos' result [VSC, Theorem, page 86], which asserts that the simple random walk on a finitely generated group is recurrent if and only if the group is virtually $0, \mathbb{Z}, \mathbb{Z}^2$ (the "if" part is a easy exercice). $\square$

Now we may proceed with the examples.

Real hyperbolic space

Let M be a smooth connected closed 3-manifold fibering over the circle. That is M is obtained by taking a smooth closed connected surface S of genus $g \geq 2$, taking its product by the interval $[0,1]$ and gluing the two boundary components $S \times \{0\}$ and $S \times \{1\}$ by a diffeomorphism ϕ. Note that one has an exact sequence $1 \to \pi_1 S \to \pi_1 M \to \mathbb{Z} \to 0$, the last map been induced by the fibration $M \to \mathbb{S}^1$ obtained by pinching the $S \times \{t\}$.

W. Thurston proved (see [Ota3]) that for a large class of such ϕ's (precisely when ϕ is homotopic to a pseudo-Anosov map), the manifold M carries a real hyperbolic metric.

Consider the Riemannian covering $\overline{M} \to M$ defined by the morphism $\pi_1 M \to \mathbb{Z}$. In particular $\pi_1 \overline{M}$ is isomorphic to $\pi_1 S$, hence is finitely generated. D. Sullivan [Sul6] proved that $\overline{M}$ carries no positive super-harmonic functions but the constants. In particular $\overline{M}$ has no finite Green function. It follows from Theorem 2.1. that $\pi_1 \overline{M}$ is a divergence group (for the real hyperbolic 3-space). Note that these groups are finitely generated Kleinian groups with infinite volume.

Let $\mathbb{F}_n$ be the free group on n generators. Let $\mathcal{R}$ be the space of representations of $\mathbb{F}_n$ in $\mathrm{PSL}_2(\mathbb{C})$. Let $\mathcal{D} \subset \mathcal{R}$ be the subset of faithful representations with discrete and convex cocompact images. Let $\partial\mathcal{D} = \overline{\mathcal{D}} - \mathcal{D}$ be the frontier of the open set $\mathcal{D}$ (whose elements are still discrete and faithful representations). Then (see [CuSh] for $n = 2$, and [ACCS] for any n), there exists a dense G_δ-subset of $\partial\mathcal{D}$ which consists of geometrically infinite divergence groups (on n generators).

There are lots of infinitely generated Fuchsian groups that are divergence groups : consider $a_i, b_i, c_i > 0$ for $i \in \mathbb{Z}$; let P_i (resp. Q_i) be the hyperbolic pair of pants with geodesic boundary of lengths a_i, b_i, c_i (resp. a_{i+1}, b_i, c_i). Glue

P_i and Q_i along their matching sides of lengths b_i, c_i, to get a surface R_i with two boundaries of lengths a_i, a_{i+1}. Then form the following "ladder" by gluing consecutively the R_i's for $i \in \mathbb{Z}$.

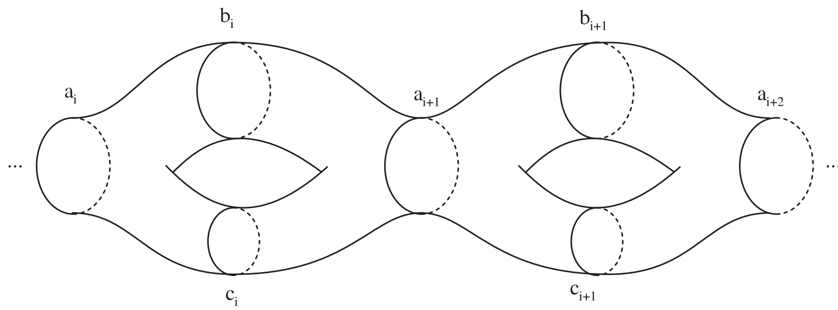

Figure 2. An hyperbolic ladder

Then if the a_i are bounded (or even not growing too fast as $i \to \pm\infty$), then the same proof by D. Sullivan shows that the Fuchsian group given by this hyperbolic ladder is a divergence one.

Complex hyperbolic 2-dimensional space

Consider F. Hirzerbruch's example Y_1 [Hir, page 134], which is a closed complex hyperbolic surface. It admits [Ish, §6, Example 5] a group G of order 125 acting freely on it, with quotient Y_1/G of Euler caracteristic 15, and with $H^1(Y_1/G, \mathbb{Z})$ infinite. Note that Y_1/G is a closed complex hyperbolic surface of smallest Euler caracteristic that we know that has a non trivial first cohomology group. (See [HP])

Let $\pi_1(Y_1/G) \to \mathbb{Z}$ be any epimorphism. Note that $\pi_1(Y_1/G)$ is a discrete cocompact isometry group of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. Then by Corollary 2.2, the kernel Γ_1 of this morphism is a divergence group. We remark that Γ_1 has infinite volume and might be not finitely generated.

Question: Is there a closed complex hyperbolic manifold fibering over the circle, or with fundamental group mapping onto $\mathbb{Z}$ with finitely generated kernel ? (This can be theorically detected by the Bieri-Neumann-Strebel invariant [BNS]).

2.2. Simplicial trees

For the trees case, the statement analogous to Theorem 2.1 has been obtained by M. Coornaert and A. Papadopoulos, with an analogous proof. We give the following special case of their result:

Theorem 2.3. *Let Γ be a subgroup of the free group of rank n, acting freely and discretely on the regular tree T_{2n} of degree $2n$. Then the simple random walk on the graph T_{2n}/Γ is recurrent if and only if Γ is a divergence group of critical exponent* $\log(2n-1)$.

Let G be the graph in Figure 3, which is the real line with a vertex at each integer point and a loop at each vertex. Its universal cover is T_4, the homogeneous simplicial tree of degree 4.

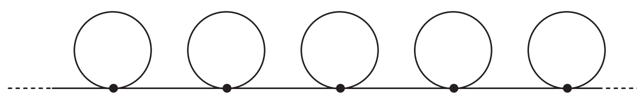

Figure 3. A graph with two ends

Let us denote by $\mathbb{F}_2 = \langle a, b\rangle$, the free group on two generators. It is well known that T_4 is isomorphic to the Cayley graph of $\mathbb{F}_2$ for the generating set $\{a, b\}$. It is clear that the graph G is isomorphic to the quotient of T_4 by the subgroup $\Gamma_{\mathbb{Z}}$ of $\mathbb{F}_2$ generated by $\{a^m b a^{-m} \,/\, m \in \mathbb{Z}\}$.

Note that the recurrence of the simple random walk is invariant by quasi-isometry (for uniformly locally finite graphs) [Kan]. Since G is quasi-isometric to $\mathbb{Z}$, the simple random walk on G is recurrent. Hence by Theorem 2.3, $\Gamma_{\mathbb{Z}}$ is a divergence subgroup of $\mathrm{Aut}(T_4)$. Since $\Gamma_{\mathbb{Z}}$ is acting freely on T_4, it has infinite covolume (in the sense of [BaKu]) and it is not finitely generated.

In Figure 4, we consider the quotient of T_4 by the group $\Gamma_{\mathbb{N}}$ generated by $a^m b a^{-m}$ for $m \in \mathbb{N}$.

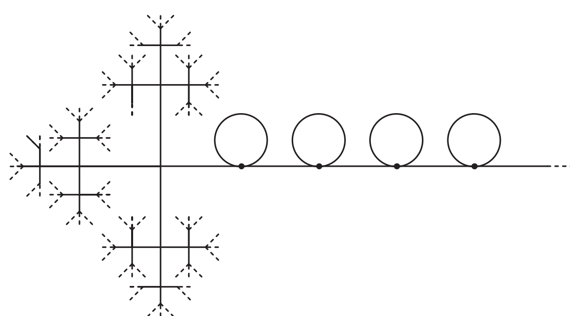

Figure 4. A graph with one geometrically infinite end

Denote by $C(\Lambda_{\Gamma_{\mathbb{N}}})$ the convex-hull of the limit set of $\Gamma_{\mathbb{N}}$. It is easy to see that the random walk on $C(\Lambda_{\Gamma_{\mathbb{N}}})/\Gamma_N$ is recurrent (as above, it is quasi-isometric to $\mathbb{N}$).

Lemma 2.4. *Let X be a proper hyperbolic metric space (in the sense of Gromov [Gro1,GH]), and let G be a discrete subgroup of isometries of X. Let X' be a*

G-invariant quasiconvex subspace (in the sense of Gromov [Gro1]). Then G is a divergence group of isometries of X if and only if it is a divergence group of isometries of X'.

Proof. For x in X, let x' in X' be a closest point to x. Then there exists a constant $A \geq 0$ (depending only on the constants of hyperbolicity of X and of quasiconvexity of X' in X) such that

$$2d_X(x,x') + d_{X'}(x',\gamma x') - A \leq d_X(x,\gamma x) \leq 2d_X(x,x') + d_{X'}(x',\gamma x') + A.$$

In particular, the Poincaré series of X at (s,x) converges if and only if the Poincaré series of X' at (s,x') converges. $\square$

By applying the above result to $X = T$ and $X' = C(\Lambda_{\Gamma_{\mathbb{N}}})$, we get that $\Gamma_{\mathbb{N}}$ is a divergence group of isometries of T_4. As above, $\Gamma_{\mathbb{N}}$ is infinitely generated and has infinite covolume.

Note that a well known result asserts that a discrete subgroup of automorphisms of a locally finite tree is convex-cocompact if and only if it is finitely generated.

2.3. Negatively curved polygonal 2-complexes

The main feature of the rank one symmetric spaces is that they have lots of symmetries, i.e. of isometries. The following family of polygonal complexes, whose study has been initiated by M. Gromov [Gro2], see also [Hag,Ben1,BaBr], is interesting precisely because (some of) these complexes have lots of symmetries.

Recall that a *Coxeter matrix* M over a set I is a symmetric matrix $(m_{ij})_{i,j\in I}$ with entries in $\mathbb{N} \cup \{\infty\}$ such that for all i, j, $m_{i,i} = 1$ and $m_{i,j} \geq 2$ for $i \neq j$. The *Coxeter group* determined by M is the group W with generators s_i for $i \in I$ and relations $(s_i s_j)^{m_{ij}} = 1$ for $i, j \in I$ with $m_{ij} \neq \infty$.

Let L be a *simplicial graph*, i.e. a finite connected graph without vertices of degree 1 or 2 and without loops or double edges. Let k be an even integer. Let $M = M(k,L)$ be the Coxeter matrix over the set of vertices of L, defined as follows. For $i \neq j$, $m_{i,j} = \frac{k}{2}$ if there exists an edge between the vertices i and j, and $m_{i,j} = \infty$ otherwise. Let $W = W(k,L)$ be the associated Coxeter group.

We define a cellular 2-complex $\Sigma = \Sigma(k,L)$ (see [Hag,Ben1,BaBr]), satisfying the following properties. The 2-cells are regular polygons with k sides. Each link of vertex in Σ is isomorphic (as graph) to L. The group W acts by automorphisms on Σ.

Let X be the cone over the barycentric subdivision L' of L, with cone point denoted by x_0. Note that X is a finite simplicial 2-complex. Let X_i be the star in L' of the vertex i of L, that we view as a subcomplex of X. Consider Σ' the quotient of $W \times X$ by the equivalence relation generated by $(w,x) \sim (ws_i, x)$ for every $i = 1 \cdots n$ and $x \in X_i$. Note that W acts on the left on Σ', by $g \cdot (w,x) = (gw, x)$

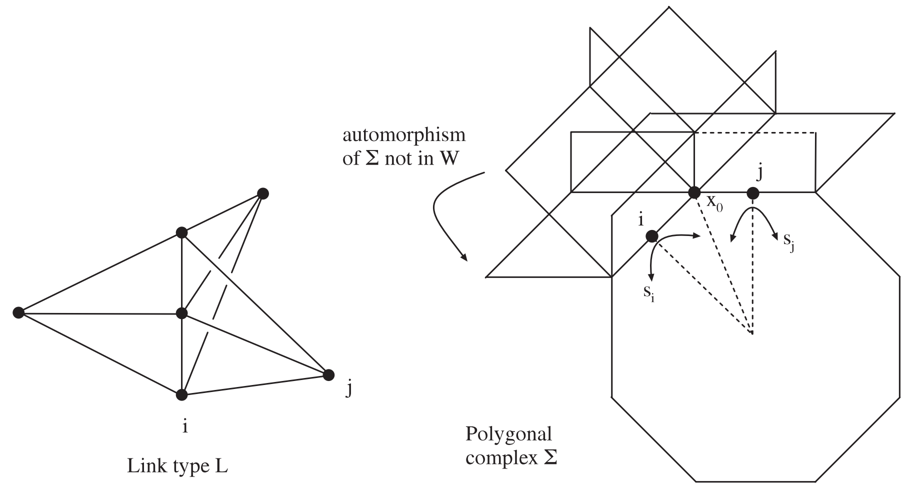

Figure 5. A CAT(−1) space with an uncountable automorphism group

Then using the relations $(s_i s_j)^{\frac{k}{2}} = 1$, it is easy to see that Σ' is the barycentric subdivision of a locally finite polygonal 2-complex Σ satisfying the required properties.

Note that X is simply connected, the X_i's are connected and for every subset S' of vertices of L, if $\bigcap_{s \in S'} X_s$ is non-empty, then the special subgroup $W_{S'}$ of W generated by S' is finite. It then follows from a result of [Dav1] that Σ is simply connected, with W acting properly discontinuously on Σ with quotient precisely X. In particular, W acts transitively on the vertices of Σ.

Assume k is bigger than 8. Identify each k-gon of Σ with a regular hyperbolic polygon with k sides (in $\mathbb{H}^2_{\mathbb{R}}$) with angles $\frac{2\pi}{3}$. This is possible since such a polygon exists for all angles α with $0 < \alpha < \frac{k-2}{k}\pi$ and $k \geq 8$. Since every simple closed loop in L has at least 3 vertices, it follows from [Gro1, 4.2.D] that Σ is locally CAT(−1). Since Σ is simply connected, it follows from [Gro1, page 119] that Σ is a proper CAT(−1) space.

The following properties of Σ may be found between the lines in [Ben1,Ben2]. F. Haglund gave us a complete proof of the last assertion (see [Hag]). It is morally the same proof that the one proving that a regular tree has an uncountable automorphism group.

Proposition 2.5. *The following properties hold.*

1. *If no edge of L separates, then $\partial\Sigma$ is arcwise connected and locally arcwise connected.*
2. *If between the endpoints of any edge e of L, there are at least two edge paths, meeting only at their endpoints and not containing e, then $\partial\Sigma$ has no local cut point (i.e. no point x such that for some neigborhood U, $U - \{x\}$ is not connected).*

3. If L has a vertex i_0 and an automorphism, different from the identity, pointwise fixing the star of i_0, then the automorphism group of Σ is uncountable. □

Note that our example in Figure 5 does satisfy all three hypotheses. Here are two more graphs also satisfying these three hypotheses. The right one is the Petersen graph P and the left one is the complete bipartite graph $K(3,3)$ on $3+3$ vertices.

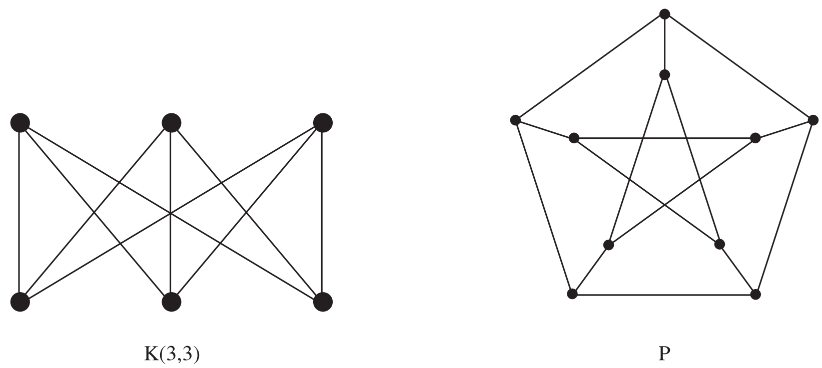

Figure 6. Flexible links

As F. Haglund told us (see [Hag]), if L is the complete graph on n vertices, then the automorphism group of Σ is countable (if an automorphism is the identity on the star of one vertex, then it is the identity, by a combinatorial analog of the analytic extension).

In particular, the examples of Figures 5, 6 give a $\partial\Sigma$ which is a metrizable compact topological space, of (topological) dimension one, arcwise connected, locally arcwise connected and without local cut point. Hence $\partial\Sigma$ is topologically a Sierpinski or Menger curve. In fact, this is the "generic" situation for the boundary of a hyperbolic group, see [Cha].

The Coxeter group W is a hyperbolic group in the sense of Gromov, since it acts cocompactly on the CAT(-1) space Σ.

If L satisfies property (1) of Proposition 2.5, then W does not decompose as a non trivial amalgamated product or an HNN extension over a finite group (otherwise its boundary would not be connected, the converse being true by Stallings theorem on ends of groups, see [GH, page 134]). If L satisfies property (2) of Proposition 2.5, then ∂W has no local cut point. Hence (see for instance [Bow]), W does not split as a non trivial amalgamated product or an HNN extension over a virtually cyclic group. In particular, $\mathrm{Out}(W)$ is finite (see for instance [Pau2]).

Note that W has no non trivial morphism to $\mathbb{Z}$, since it is generated by torsion elements. (In particular, this also proves that W is not an HNN extension). But there often exist finite index (torsion free) subgroups of W having morphisms onto $\mathbb{Z}$.

Lemma 2.6. *For $L = K(3,3)$ and $k \geq 8$ even, or for L which is not the complete graph, and $k \geq 8$ a multiple of 4, the word hyperbolic Coxeter group $W(k,L)$ has non zero virtual first Betti number.*

Proof. In the first case, the group W admits the following presentation:

$$\langle a_1, a_2, a_3, b_1, b_2, b_3 / a_i^2 = b_i^2 = (a_i b_j)^{k/2} = 1\rangle.$$

Consider the epimorphism π from W onto the triangle group with presentation $\langle a_1, a_2, a_3 / a_i^2 = (a_i a_j)^{k/2} = 1\rangle$, obtained by adding the relations $a_i = b_i$.

Now this triangle group has a finite index subgroup G which is the fundamental group of a connected closed orientable surface of genus at least 1. Take $\Gamma = \pi^{-1}(G)$, which has finite index in W. Then by composing the restriction of π to Γ with an epimorphism $G \to \mathbb{Z}$, the result follows. (Except that $W(k,L)$ is no longer word hyperbolic, this also work for $k = 6$, as pointed out by the referee.)

For the second case, consider vertices i, j in L which are not joined by an edge. Consider the epimorphism π from W to the free product of two order two groups $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ with generators a_i, a_j, obtained by mapping every generator of W except a_i, a_j to the identity. Since k is a multiple of 4, and i, j are not joined by an edge, the relations $(a_k a_l)^{k/2}$ are mapped to relations that are consequences of $a_i^2 = 1, a_j^2 = 1$, and π is indeed well defined.

Now the group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ contains an infinite cyclic subgroup $\mathbb{Z}$ with index 2, and the result follows as above. $\square$

Note that in the second case, the kernel of the morphism π is the normal closure of the special subgroup of W generated by all generators of W except a_i, a_j, which is unfortunately in general not finitely generated.

2.4. Hyperbolic Bruhat-Tits buildings

The CAT(-1) complex $\Sigma(k,L)$ of the previous subsection for $L = K(3,3)$, with 2-cells the regular hyperbolic k-gon with angles $\frac{\pi}{2}$, has the following striking properties:

- it is the unique simply connected polygonal 2-complex whose 2-cells are k-gons and links of vertices isomorphic to $K(3,3)$ ([Swi] Theorem 0.6 (1));
- its group of automorphisms is an uncountable locally compact group, by Proposition 2.5 (and Theorem 0.6 (1) of [Swi]);
- Any Möbius map from the boundary of Σ to any locally compact CAT(-1) space is the extension of an isometric embedding (see [Bou4], who also computes the conformal dimension (in Pansu's sense) of $\partial\Sigma$);
- it is an hyperbolic Bruhat-Tits building whose type is the (infinite) Coxeter group generated by the hyperbolic reflections on the sides of a regular hyperbolic k-gon with angles $\frac{\pi}{2}$ (in the sense we define below).

We recall some facts from [Ron] Chapters 1,2,3 (see also [Dav2]).

A *chamber system* over a set I is a set C together with a family of partitions of C indexed by I. The elements of C are called *chambers*. Two chambers are i*-adjacent* if they belong to the same subset in the partition corresponding to i.

Let C, D be chamber systems over I. A *morphism* $f : C \to D$ is a map such $f(c), f(c')$ are i-adjacent that for all i-adjacent chambers $c, c' \in C$. We denote by $\mathrm{Aut}(C)$ the group of automorphisms of a chamber system C.

Examples. (1) Let G be a group, B a subgroup, $(P_i)_{i\in I}$ a family of subgroups containing B. Define the chamber system $C = C(G, B, (P_i)_{i\in I})$ by $C = G/B$ and the chambers $gB, g'B$ are i-adjacent if they have the same image in G/P_i.

The group G is naturally a chamber transitive group of automorphisms of C. The kernel of the action of G on C is $\bigcap_{g\in G} gBg^{-1}$.

(2) Let M be a Coxeter matrix over a set I, and W the associated Coxeter group. For $J \subset I$, let W_J be the subgroup of W generated by $\{s_j\}_{j\in J}$. Let $\mathbf{W}$ be the chamber system $C(W, \{1\}, (W_{\{i\}})_{i\in I})$ over I, called the *Coxeter system* of type M. Thus, the set of chambers is W, and two chambers w, w' are i-adjacent if and only if $w' = w$ or $w = w's_i$.

Let M be a Coxeter matrix over a set I. A *(Bruhat-Tits) building* of type M is (see [Ron, page 34]) a chamber system C over I, endowed with a maximal family of subsystems (called *apartments*), isomorphic to the Coxeter system of type M, such that

1. any two chambers lies in a common apartment,
2. given two apartments A, A' containing a common chamber c and chamber or panel y, there is an isomorphism of C fixing x, y and sending A to A'.

Let C, D be buildings of type M. A *morphism* $f : C \to D$ is a morphism of the underlying chamber systems.

Example (3). (Meier-Davis) Let L be a simplicial graph with vertex set I. Let $k = 4$. Consider the (right angled) Coxeter matrix $M = M(k, L)$ defined in the previous subsection. Let $(P_i)_{i\in I}$ be a family of groups indexed by I. Let G be the *graph product of groups* of $(P_i)_{i\in I}$, that is the group generated by P_i for $i \in I$, with relations $[g_i, g_j] = 1$ for all $g_i \in P_i, g_j \in P_j$ and $m_{ij} = 2$ (i.e. there is an edge between i, j).

Then by [Dav2, Theorem 5.1], $C = C(G, \{1\}, (P_i)_{i\in I})$ is a building of type M.

Let C be a chamber system over I. A *gallery* is a finite sequence of chambers $(c_0, c_1, \cdots, c_n)$ such that c_j is adjacent but not equal to c_{j-1} for $j = 1 \cdots n$. Let I^* be the free monoid on I. The gallery has *type* $\tau = i_1 \cdots i_n \in I^*$ if c_j is i_j-adjacent to c_{j-1}. The *rank* of C is the cardinal of I.

Let $J \subset I$. A chamber system over I is J*-connected* if any two chambers c, c' can be joined by a gallery of type $\tau = i_1 \cdots i_n$ with $i_j \in J$. The J-connected components of C are called the J*-residues* of C.

Let C be a building of type M. A subset J is called *spherical* if W_J is finite. Let $\mathcal{S}^f$ be the set of spherical subsets of I, partially ordered by the inclusion.

Recall that the *derived complex* of a partially ordered set P is the set of finite chains in P, partially ordered by inclusion. It is an abstract simplicial complex, with vertex set P. Its geometric realization is called the *geometric realization* of P.

The *geometric realization* of C (see [Dav2, §9]), denoted by $|C|$, is the geometric realization of the set of all J-residues in C with $J \in \mathcal{S}^f$, partially ordered by inclusion.

The following theorem is one of the main source of CAT(-1) spaces with big automorphism groups.

Theorem 2.7. (Moussong-Davis [Mou] [Dav2, Remark 11.10]) *Let M be a Coxeter matrix over I, with I finite. Assume that for every subset J of I, neither of the following occurs:*
1. W_J is of affine type (see [Ron, Chapter 9]) with Card$(J) \geq 3$
2. $J = J_1 \cup J_2$ and W_J is the direct product of $W_{J_1} \times W_{J_2}$ with W_{J_1}, W_{J_2} infinite.
Let C be a building of type M. Then $|C|$ admits a (piecewise hyperbolic) CAT(-1) metric such that Aut(C) acts by isometries.

The CAT(-1) space $|C|$ as in this theorem will be called a *hyperbolic (Bruhat-Tits) building*. Typical examples are those for W a discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$, generated by the reflections on the faces of a compact convex polyhedra in $\mathbb{H}^n_{\mathbb{R}}$ (with diedral angles of the form π/k, with $k \in \mathbb{N}, k \geq 2$). Even if, as shown by E. Vinberg [Vin], these exist only in dimension less than 29, this class of CAT(-1) spaces is highly interesting.

Let X be a hyperbolic building of the form $|C|$ with $C = C(G, B, (P_i)_{i\in I})$. Then G acts faithfully if and only if $\bigcap_{g\in G} gBg^{-1} = \{1\}$. The space X is locally compact if and only if B has finite index in P_i for each i. If X is locally compact, G is a discrete group of isometries of X if and only if B is finite. Since G is chamber transitive, it is cocompact.

Example (4). (J. Meier [Mei]) Let L be the circuit of length k. Let $(P_i)_{i=1\cdots k}$ be k copies of the cyclic group $\mathbb{Z}/3\mathbb{Z}$. Let G be the associated graph product of groups. Consider the hyperbolic building $C = C(G, \{1\}, (P_i)_{i=1\cdots k})$ of type the Coxeter group generated by reflections on the sides of a regular hyperbolic k-gons of angles $\frac{\pi}{2}$. Then M. Bourdon [Bou4] remarked that C is isometric to $\Sigma(k, K(3,3))$. This follows from an easy computation of the links of vertices, by the uniqueness property of $\Sigma(k, K(3,3))$.

If X is a locally finite hyperbolic building, we can define a Laplacian operator and Brownian motion on Σ, by saying that the Laplace operator inside a chamber is precisely the Riemannian one, and that a Brownian path hitting a panel (i.e. a codimension one face) has equiprobability to keep on moving in one of the adjacent cells. (The probability of hitting the codimension 2 skeleton is 0.) See for instance

[BrKi] in the case of polygonal 2-complexes.

Say that a building C is *2-point transitive* if its (possibly type rotating) automorphism group acts transitively on the pair of chambers having same combinatorial distance. So the following analog of Theorems 2.1 and 2.3 holds.

Proposition 2.8. *Let X be a 2-point transitive locally compact hyperbolic building. Let Γ be a discrete group of isometries of X. Then the following are equivalent:*
1. *the Brownian motion on $M = X/\Gamma$ is recurrent*
2. *M has no (finite) Green function*
3. *Γ is a divergence group and the critical exponent of Γ is the volume entropy of X*

$$h_X = \limsup_{k\to\infty} \frac{1}{k} \log \mathit{Vol}\, B_X(x,k)$$

where $B_X(x,k)$ is the ball of radius k and center any x in X. □

By this proposition, example (4), Lemma 2.6, and the same arguments as in the proof of Corollary 2.5, one has a precise example:

Corollary 2.9. *For $L = K(3,3)$ and $k \geq 8$ even, there exists an epimorphism from a finite index subgroup of $W(k,L)$ onto $\mathbb{Z}$, whose kernel Γ is a divergence group of isometries of the locally compact hyperbolic building $\Sigma(k,L)$.* □

Note that the convex hull in $\Sigma(k,L)$ of the limit set of Γ (which is the whole boundary) has infinite covolume, hence Γ is far from being convex-cocompact.

We state here a result that will be useful in the next section.

Lemmma 2.10. *Let X be a building with Coxeter group W. Let $x \in X$ be an interior point of a chamber. For any two apartments A, B containing x, there exists a sequence of apartments $(A_n)_{n\in\mathbb{N}}$ with $A_0 = A$, such that $A_n \cap A_{n+1}$ contains an half-apartment containing x in its interior, and such that (A_n) converges to B for the topology of uniform convergence on compact subsets.*

Proof. We may assume $A \neq B$. Let $A_0 = A$. The closure of the interior of $A_0 \cap B$ is a convex union of chambers (see [Ron, Theorem 3.8, p. 33]). Let σ be a chamber of B, meeting A_0 in exactly a codimension one face F_i. Let H be the half-apartment of A_0 containing $A_0 \cap B$, whose wall contains F_i (see [Ron, p. 13-14] for the definitions).

Then $H \cup \sigma$ is clearly W-isometric (see [Ron, p. 31]) to a subset E of $|W|$. The W-isometry $E \to H \cup \sigma \subset X$ extends to a W-isometry $|W| \to X$. Its image is by maximality an apartment A_1. Note that $A_0 \cap A_1$ contains an half-apartment containing x in its interior. Also note that $A_1 \cap B$ contains at least one more chamber than $A_0 \cap B$.

Since an apartment is a locally finite union of chambers, the construction of $\{A_n\}$ (possibly stationary) is clear by induction. By choosing suitably the σ's, we may assume that $A_n \cap B$ contains a ball of center x and radius tending to ∞ as $n \to \infty$. Hence the result follows. □

3. Ergodic rigidity of CAT(-1) discrete groups of isometries

This section is devoted to the proofs of Theorem A and Corollary B.

We fix the notations. Let X_1, X_2 be proper CAT(-1) spaces, and respectively:

- Γ_1, Γ_2 discrete subgroups of isometries of X_1, X_2,
- x_1, x_2 base points in X_1, X_2,
- δ_1, δ_2 the critical exponents of Γ_1, Γ_2, assumed to be neither 0 nor ∞,
- $[\cdot,\cdot,\cdot,\cdot]_1, [\cdot,\cdot,\cdot,\cdot]_2$ the crossratios on $\partial X_1, \partial X_2$,
- μ_1, μ_2 the Patterson-Sullivan measures on $\partial X_1, \partial X_2$, viewed from x_1, x_2
- ν_1, ν_2 the Bowen-Margulis measures on $\partial X_1 \times \partial X_1 - \Delta, \partial X_2 \times \partial X_2 - \Delta$.

A map $\tilde{\phi} : \partial X_1 \to \partial X_2$ is *almost everywhere Möbius* if

$$[\tilde{\phi}a, \tilde{\phi}b, \tilde{\phi}c, \tilde{\phi}d]_2 = [a, b, c, d]_1$$

holds for distinct a, b, c, d in ${\partial X_1}^4$ outside a set of measure 0 for μ_1^4. Note that if $\tilde{\phi}$ is absolutely continuous for the Patterson-Sullivan measures, then $\tilde{\phi}a, \tilde{\phi}b, \tilde{\phi}c, \tilde{\phi}d$ are distinct for almost every a, b, c, d. The following generalizes [Sul4, Theorem 5] to our settings, and its proof follows closely Sullivan's approach.

Proposition 3.1. *Suppose that $\tilde{\phi} : \partial X_1 \to \partial X_2$ is a Borel map such that $\tilde{\phi} \times \tilde{\phi}$ preserves the Bowen-Margulis measures up to a constant. If $\delta_1 = \delta_2$, then $\tilde{\phi}$ coincides, almost everywhere for the Patterson-Sullivan measure, with a Möbius map on the limit set of Γ_1.*

*Proof.*Note that by disintegration, $\tilde{\phi}$ is absolutely continuous for the Patterson-Sullivan measures. (In the application, this will be an hypothesis.)

Let us remark that if m, m' are measures on a Borel space B, that are in the same class, and if $f : B \to C$ is a Borel map, then f_*m, f_*m' are in the same class and $\frac{d(f_*m)}{d(f_*m')}(f(b)) = \frac{dm}{dm'}(b)$ for almost every b in B.

For ν_1-almost every (a, b) in $\partial X_1 \times \partial X_1 - \Delta$, with $t > 0$ some constant, we have by hypothesis:

$$\frac{d\,\tilde{\phi}_*\mu_1 \times d\,\tilde{\phi}_*\mu_1(\tilde{\phi}a, \tilde{\phi}b)}{d_{x_1}(a, b)^{2\delta_1}} = t\frac{d\,\mu_2 \times d\,\mu_2(\tilde{\phi}a, \tilde{\phi}b)}{d_{x_2}(\tilde{\phi}a, \tilde{\phi}b)^{2\delta_2}}.$$

Therefore, for ν_1-almost every (a, b), we have:

$$\frac{d_{x_1}(a,b)^{2\delta_1}}{d_{x_2}(\tilde{\phi}a, \tilde{\phi}b)^{2\delta_2}} = t\frac{d\,\tilde{\phi}_*\mu_1}{d\,\mu_2}(\tilde{\phi}a)\frac{d\,\tilde{\phi}_*\mu_1}{d\,\mu_2}(\tilde{\phi}b).$$

Since the right handside of this equation is a separable function in a, b, an easy computation using (7), shows that

$$\delta_2[\tilde{\phi}a, \tilde{\phi}b, \tilde{\phi}c, \tilde{\phi}d]_2 = \delta_1[a, b, c, d]_1$$

for a, b, c, d in ${\partial X_1}^4$ outside a set of measure 0 for μ_1^4. If $\delta_1 = \delta_2$, then the crossratios are almost everywhere preserved.

Now, if a map is Möbius on a subset of ∂X_1, then it is uniformly continuous on this subset, for instance because it is an homothety for the distances constructed in the Appendix, see Corollary A.3. Since the support of the Patterson-Sullivan measure μ_1 is the limit set of Γ_1, the map $\tilde{\phi}$, being uniformely continuous on a dense subset, can be continuously extended to a map on the whole limit set. The extension is clearly Möbius, by continuity of the crossratio in its four variables. □

Proof of Theorem A. It was showed in [BuMo, section 6] that the divergence of the Poincaré series of Γ_2 implies that the diagonal action of Γ_2 with respect to the measure ν_2 is ergodic. (Their proof closely follows the one for the real hyperbolic case, given by D. Sullivan [Sul1,Sul2].)

Recall that push-forwards of measures preserves the nonsingularity. Since $\tilde{\phi}$ is absolutely continuous for the Patterson-Sullivan measures, then $\tilde{\phi}\times\tilde{\phi}$ is absolutely continuous for the Bowen-Margulis measures.

By equivariance of $\tilde{\phi}$, the measurable function

$$\frac{d\,(\tilde{\phi}\times\tilde{\phi})_*\nu_1}{d\,\nu_2}$$

is invariant by the diagonal action of Γ_2 on $\partial X_2 \times \partial X_2$. Using the ergodicity of Γ_2, we deduce that $(\tilde{\phi}\times\tilde{\phi})_*\nu_1 = t\nu_2$ for some positive constant t.

We end the proof by applying Proposition 3.1. □

Let us now specialize our CAT(-1) space X_1.

*Proof of Corollary B.*We have to prove that in the special situations of the corollary, a Möbius embedding of ∂X_1 in ∂X_2 is the extension of an isometric embedding of X_1 in X_2.

(1) Asumme X_1 is a rank one symmetric space $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{H}}$, $\mathbb{H}^2_{\mathbf{C}a}$, normalized so that its maximal curvature is precisely -1. The result follows precisely from M. Bourdon's result [Bou3], that a Möbius embedding of ∂X_1 is the extension of an isometric embedding. (See [Yue] if X_2 is also a rank one symmetric space).

(2) Let $\tilde{\phi} : \partial X_1 \to \partial X_2$ be a Möbius embedding, with X_2 a CAT(-1) space and X_1 a (locally compact) hyperbolic Bruhat-Tits building modeled on a discrete cocompact reflection group W of $\mathbb{H}^n_{\mathbb{R}}$. Let us construct an isometric map $\phi : X_1 \to X_2$ whose extension to the boundary is $\tilde{\phi}$. By density, it is sufficient to construct ϕ on the union of the interiors of chambers.

Let x be an interior point of a chamber. Let A be an apartment containing x.

Since A is isometric to $\mathbb{H}^n_{\mathbb{R}}$ (of constant curvature -1), the Möbius embedding $\tilde{\phi}|_{\partial A}$ from ∂A into ∂X_2 extends to an isometric embedding $\phi_A : A \to X_2$ (see [Bou3, Théorème 0.1]). Recall that the map ϕ_A is defined as follows. Let $x \in A$, let g, g' be (any) geodesics in A meeting in $\{x\}$, with endpoints a, b and a', b' respectively. Then the geodesics with endpoints $\tilde{\phi}(a), \tilde{\phi}(b)$ and $\tilde{\phi}(a'), \tilde{\phi}(b')$ meet in one and only one point, $\phi_A(x)$.

We claim that $\phi_A(x)$ does not depend on A. Indeed, let B be another apartment containing x. Let A_n be a sequence of apartments as in Lemma 2.10. Since $A_n \cap A_{n+1}$ contains an open half-apartment containing x, the point x is the intersection of a pair of geodesics lying in both A_n and A_{n+1}. Hence $\phi_{A_n}(x) = \phi_{A_{n+1}}(x)$. Since (A_n) converges to B for the uniform convergence on compact subsets, we have that (∂A_n) converges to ∂B for the Hausdorff distance on ∂X_1 (endowed with any visual metric). By continuity of $\tilde{\phi}$, we have $\phi_A(x) = \phi_B(x)$.

Hence we get a well defined map $\phi : X_1 \to X_2$. Since any two points of X_1 belong to an apartment, and since the restriction of ϕ to any apartment is an isometry, the map ϕ is an isometry. This proves the result. □

Here is another example where the above proof (part (2)) applies. Let X be a connected compact hyperbolic surface with one boudary component which is totally geodesic. Take $n \geq 2$ copies of A, and glue them isometrically along their boundary. Denote by X_1 the resulting space. Then with this X_1, the conclusion of Theorem A can also be strengthened as above.

Here is an example of two CAT(-1) spaces X_1, X_2 with a Möbius homeomorphism $\tilde{\phi} : \partial X_1 \to \partial X_2$, which is not the extension of an isometry, where X_1 is a tree (with no terminal vertex).

Let X_1 be the regular tree of degree 4. Let $\epsilon \in]0, \frac{1}{2}]$. Let $V_\epsilon(x)$ be the ϵ-neighborhood of a vertex x of X_1, which has four terminal vertices. Let T_ϵ be the regular (real) hyperbolic tetrahedra with edge length 2ϵ. Note that one has a continuous map $T_\epsilon \to V_\epsilon(x)$ which is an isometry on each edge of T_ϵ. Remove all ϵ-neighborhoods of vertices in X_1, and glue, for each vertex x of X_1, a copy of T_ϵ, by identifying a vertex of T_ϵ to the point of $X_1 - \bigcap_{x \in X_1^{(0)}} V_\epsilon(x)$ corresponding to a terminal vertex of $V_\epsilon(x)$. The resulting space X_2 (uniquely defined up to isometry) is CAT(-1). One has a map $X_2 \to X_1$ which pinches each copy of T_ϵ to $V_\epsilon(x)$ by the above map. This map induces an homeomorphism $\partial X_2 \to \partial X_1$ whose inverse we denote by $\tilde{\phi}$. It is easy to see that $\tilde{\phi}$ is Möbius. But there is no isometric embedding of X_1 into X_2.

In the case where Γ_1, Γ_2 are cocompact, we can obtain more information.

Corollary 3.2. *Assume that Γ_1, Γ_2 are cocompact, and let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphism.*

1. *There exists a ρ-equivariant homeomorphism $\tilde{\phi} : \partial X_1 \to \partial X_2$ (called the boundary map).*
2. *If $\tilde{\phi}$ is non-singular with respect to the Patterson-Sullivan measures, and the critical exponents δ_1, δ_2 are equal, then $\tilde{\phi}$ is Möbius (everywhere).*

Proof. The existence of $\tilde{\phi}$ follows from the fact that Γ_1, Γ_2 are word hyperbolic in the sense of Gromov. Their limit sets can be identified with their boundaries as hyperbolic groups. Since ρ is a quasi-isometry with respect to the word metric, it can be extended uniquely to an homeomorphism from the limit set of Γ_1 to the limit set of Γ_2 (see for instance [GH]).

The result now follows easily, since the limit sets are the whole boundaries in this case. □

4. Marked length spectrum rigidity for negatively curved surfaces with singularities

A *negatively curved cone surface* is a surface M endowed with a *negatively curved cone metric*, i.e. a smooth negatively curved Riemannian metric on $M - P$, where P is a discrete subset of points of M, such that the completion of $M - P$ is M, and such that the (obviously defined) *cone angle* at each singularity is $\geq 2\pi$.

For example, branched covers of closed negatively curved Riemannian surfaces are negatively curved cone surfaces (with angle $2\pi n$ at each branch point of index n).

Note that the assumption on the cone angles implies that the completed distance on M is locally CAT(-1), hence that the universal cover of X is CAT(-1). Also note that ∂X is homeomorphic to a circle, with $X \cup \partial X$ homeomorphic to the closed 2-disc.

Let M be a compact locally CAT(-1) space. Let $\mathcal{C} = \mathcal{C}(\pi_1 M)$ be the set of non trivial conjugacy classes in $\pi_1 M$ (i.e. the set of non trivial free homotopy classes of closed loops in M). It is well known and easy to prove that any closed loop, which is not freely homotopic to a point, is freely homotopic to a unique closed geodesic. The map from $\mathcal{C}$ to $\mathbb{R}^+$ which associates to a non trivial conjugacy class $\langle\gamma\rangle$, the length of the unique corresponding closed geodesic c_γ, is called the *marked length spectrum.*

This section is devoted to the proof of our second main result Theorem C, asserting the equivalence between four assertions (1)-(4).

Note that (1) implies (2) has been proved in Theorem A. Clearly (4) implies (1), (2), (3). Hence we will only have to prove in what follows that (2) and (3) are equivalent (subsection 4.1) and that (2) and (3) imply (4) (subsection 4.4). Note that if there are no singularities, (3) implies (4) is due to J.-P. Otal [Ota1].

We will first give the main tools needed there. These are three measures, the brand new Möbius measure (subsection 4.1), the suitable modification of the Liouville measure (subsection 4.2), which are both geodesic currents in the sense of F. Bonahon, and the suitable modification of the Lebesgue measure on the (almost everywhere) unit tangent bundle (subsection 4.3).

4.1. The Möbius current of a CAT(-1) surface

Let M be a connected compact locally CAT(-1) space, X the universal cover of M, Γ its covering group, so that $M = X/\Gamma$.

We are first going to recall the definition of a geodesic current, developped by K. Sigmund, D. Sullivan and F. Bonahon [Bon1,Bon2,Bon3], to which we refer for basic properties and historical remarks.

Let $G(X)$ be the space of unpointed unoriented geodesics of X endowed with the topology of the Hausdorff distance on compacts set of X (i.e. two geodesics are close if their intersection with some big compact subset of X are close for the Hausdorff distance). The group Γ naturally acts on $G(X)$.

Notation. If I is a geodesic segment of X, we will denote by $\mathcal{G}(I)$ the compact subset of geodesics of $G(X)$ meeting I.

Let $\partial^2 X = \partial X \times \partial X - \Delta$, where Δ is the diagonal. Then $\partial^2 X$ is a locally compact space, endowed with the diagonal action of the group Γ, and with the action of $\mathbb{Z}/2\mathbb{Z}$ permuting the factors, commuting with the previous action. Note that $G(X)$ is Γ-equivariantly homeomorphic to (and will be identified with) $\partial^2 X/(\mathbb{Z}/2\mathbb{Z})$.

Definition 4.1. A *geodesic current* for M is a positive regular Borel measure on $G(X)$ that is invariant under the action of Γ.

Note that the topological space $G(X)$, endowed with the action of Γ, depends only on the group Γ. Hence so does the space of geodesic currents for M. Indeed, the boundary of X is Γ-equivariantly homeomorphic to the boundary of the word hyperbolic group Γ, endowed with any fixed set of generators.

Since M is compact, any $\gamma \in \Gamma$ is an hyperbolic isometry of X. Hence γ has a translation axis A_γ, on which γ acts by a translation of length $\ell(\gamma)$. Let I be a fundamental domain in A_γ for the action of γ. Let $\langle\gamma\rangle$ be the conjugacy class of γ in Γ. Let μ be a geodesic current.

Definition 4.2. The *intersection number* of μ and $\langle\gamma\rangle$ is

$$i(\mu, \langle\gamma\rangle) = \mu(\mathcal{G}(I)).$$

The fact that this intersection number is well defined (does not depend on the element of the conjugacy class $\langle\gamma\rangle$ nor on the fundamental arc I) follows from the invariance of μ under Γ.

It may also be shown (see [Bon1, Chapter 4]) that the intersection numbers depend only on the fundamental group Γ, and not on the locally CAT(-1) metric on M.

In what follows, we *assume that the boundary of X is a circle.* (Note that by the theorems of D. Gabai [Gab], A. Casson-D. Jungreiss [CJ], this is essentially the same as assuming that M is a surface, but we will not use that.)

Recall the following fundamental result of J.-P. Otal (stated only for negatively curved closed surfaces, but the proof extends easily to our situation).

Theorem 4.3. (J.-P. Otal [Ota1] Théorème 2) *A geodesic current on M is caracterized by its intersection numbers with all conjugacy classes in Γ.* □

Let us now define the Möbius current. Let $I, J \subset \partial X$ be two non-empty intervals with disjoint closures, and with set of endpoints respectively $\{a, b\}, \{c, d\}$. Define

$$\mu(I \times J) = |[a, b, c, d]|$$

where $[a, b, c, d]$ is the crossratio of the four points, if these are distinct, and $\mu(I \times J) = 0$ if I or J is reduced to a singleton. By the symmetries of the crossratios (see section 1), this depends indeed only on I, J.

Theorem 4.4. *The map μ uniquely extends to a σ-finite, regular Borel measure on $\partial^2 X$. This measure is invariant under the involution on $\partial^2 X$, hence induces a σ-finite, regular Borel measure on $G(X)$, denoted by $\mu_{M\ddot{o}b}$. Furthermore, $\mu_{M\ddot{o}b}$ is a geodesic current, and will be called the Möbius current of M.*

*Proof.*A product $I \times J$, with $I, J \subset \partial X$ non-empty intervals with disjoint closures, will be called a *rectangle*. Let $\mathcal{A}_0$ be the algebra of subsets of $\partial^2 X$ generated by the rectangles. Let us first check that μ extends to $\mathcal{A}_0$, and is finitely additive on $\mathcal{A}_0$. Let $I \times J$ be a rectangle, with set of endpoints of I, J respectively $\{a, b\}, \{c, d\}$. Assume that J is the disjoint union of two intervals J^-, J^+ whose closure meet in $\{e\}$. Upon applying equation (6) or (7), a simple cancellation argument shows that $[a, b, c, e] + [a, b, e, d] = [a, b, c, d]$. Furthermore, $[a, b, c, e], [a, b, e, d], [a, b, c, d]$ have the same sign. By induction, this proves the finite additivity of μ on $\mathcal{A}_0$.

We are going to use Carathéodory's construction to extend μ. Let us define a map $\mu^* : \mathcal{P}(\partial^2 X) \to [0, +\infty]$ by

$$\mu^*(A) = \inf\{\sum_{i \in I} \mu(R_i)\}$$

where $(R_i)_{i \in I}$ ranges over finite or countable coverings of A by open rectangles. It is easy to check that μ^* is an *outer measure*, i.e. that $\mu^*(\emptyset) = 0$, that $\mu^*(A) \leq$

$\mu^*(B)$ if $A \subset B$, and that $\mu^*(\bigcup_{i\in I} A_i) \leq \sum_{i\in I} \mu^*(A_i)$ for any finite or countable index set I. A subset A of $\partial^2 X$ is *μ^*-measurable* if

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for every subset E of $\partial^2 X$, where A^c is the complement of A. Recall that the set $\mathcal{A}$ of μ^*-measurable subsets of $\partial^2 X$ is a σ-algebra, on which μ^* is a σ-additive positive measure (see for instance [Coh, Theorem 1.3.4]). Let us now prove that the Borel σ-algebra of $\partial^2 X$ is contained in $\mathcal{A}$, and that μ^* coincides with μ on each rectangle. This will prove the result, the invariance by the involution and the isometries being obvious. We start with a "continuity lemma":

Lemma 4.5. *For every $m > 0$ and small enough $\epsilon > 0$, there exists $\eta > 0$ such that if I, J are disjoint intervals in ∂X, with endpoints a, b and c, d respectively, such that $d_x(a,b) \leq \eta$, and $d_x(a,c), d_x(a,d) \geq m$, then $\mu(I \times J) \leq \epsilon$.*

Proof. By the triangle inequality and equation (7),

$$[a,b,c,d] = \log \frac{d_x(b,d)}{d_x(a,d)} - \log \frac{d_x(b,c)}{d_x(a,c)} \leq \log \frac{d_x(b,a)+d_x(a,d)}{d_x(a,d)} - \log \frac{d_x(b,a)-d_x(a,c)}{d_x(a,c)}$$

and similarly for a lower bound. Hence $[a,b,c,d] = O(d_x(a,b))$ as $d_x(a,b)$ tends to 0, where O depends only on a positive lower bound on $d_x(a,c), d_x(a,d)$. □

Endow the subspace $\partial^2 X \subset \partial X \times \partial X$ with the supremum metric. In particular, the (open) ϵ-neighborhood of a rectangle is an open rectangle, and the ϵ-neighborhood of the boundary of A_n is in the algebra $\mathcal{A}_0$ (at least for ϵ small enough). The above lemma easily implies that for every rectangle A, the μ-mass of the ϵ-neighborhood of ∂A tends to 0 as ϵ goes to 0.

Claim 1. Every Borel subset of $\partial^2 X$ is μ^*-measurable.

Since the open rectangles generate the Borel σ-algebra, one only needs to check that every open rectangle A is μ^*-measurable. Let $E \subset \partial^2 X$. If $\mu^*(E) = \infty$, there is nothing to check, hence we may assume that $\mu^*(E) < +\infty$. For $\epsilon > 0$, let $(R_i)_{i\in I}$ be a finite or countable covering of E by open rectangles, such that $\sum_{i\in I} \mu(R_i) < \mu^*(E) + \epsilon$. The intersections $A \cap R_i$ are open rectangles that cover $E \cap A$, hence

$$\mu^*(E \cap A) \leq \sum_i \mu(R_i \cap A).$$

For $\delta > 0$, let A_δ be the union of A^c and $V_\delta(\partial A)$ (the open δ-neighborhood of the boundary of A). It follows by the finite additivity of μ on $\mathcal{A}_0$ and Lemma 4.5, that if δ_i is small enough, then $R_i \cap A_{\delta_i}$, which is in $\mathcal{A}_0$, can be written as the union of finitely many open rectangles R_i^j, say $j \in J_i$, such that $\sum_{j \in J_i} \mu(R_i^j) \leq \mu(R_i \cap A^c) + \frac{\epsilon}{2^{i+1}}$. Since the open rectangles $(R_i^j)_{i,j}$ cover $E \cap A^c$, one has

$$\mu^*(E \cap A^c) \leq \sum_{i,j} \mu(R_i^j) \leq \sum_i \mu(R_i \cap A^c) + \epsilon.$$

Hence by the above two equations and finite additivity

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \sum_i \mu(R_i) + \epsilon \leq \mu^*(E) + 2\epsilon.$$

Letting ϵ go to 0, this proves Claim 1.

Claim 2. μ^* and μ coincide on the rectangles.

Let A be a closed rectangle. Since $V_\delta(A)$ is an open rectangle for $\delta > 0$ small enough, and since it contains A, and since its μ-mass tends to the one of A, one has $\mu^*(A) \leq \mu(A)$. Conversely, for every $\epsilon > 0$, let $(R_i)_{i \in I}$ be a finite or countable covering of A by open rectangles, such that $\sum_{i \in I} \mu(R_i) < \mu^*(A) + \epsilon$. Since A is compact, we can extract a finite covering $(R_j)_{j \in J}$. By finite (sub-)additivity, one has

$$\mu(A) \leq \sum_{j \in J} \mu(A \cap R_j) \leq \sum_{j \in J} \mu(R_j) \leq \sum_{i \in I} \mu(R_i) \leq \mu^*(A) + \epsilon.$$

Since this holds for every $\epsilon > 0$, Claim 2 follows. This ends the proof of 4.4. □

Let us give a non-trivial example of a Borel subset of $G(X)$ whose Möbius current is 0. As pointed out by the referee, it follows from the easily proven fact that the support of the Möbius current is the Borel subset of endpoint sets of non singular geodesics.

Lemma 4.6. *Let $x \in X \cup \partial X$. Let $\mathcal{G}(x)$ be the set of geodesics in $G(X)$ passing through x. Then*

$$\mu_{M\ddot{o}b}(\mathcal{G}(x)) = 0.$$

Proof. If $x \in X$, this follows from the fact that, for every distinct $a, b, c, d \in \partial X$, if the four geodesics, respectively between a, b, between a, d, between b, c and between b, d, intersect in a common point u, then by its definition in equation (6), the crossratio $[a, b, c, d]$ is 0.

If $x \in \partial X$, this follows from the fact that the crossratio $[a, b, c, d]$ defined on quadruples of distinct points of ∂X can be continuously extended to quadruples (a, a, c, d) with a, c, d distinct by

$$[a, a, c, d] = 0.$$

□

The following proposition, using arguments of [Ota2], gives an important connection between the marked length spectrum and the Möbius current. This connection is essential in what follows.

Proposition 4.7. *For any $\gamma \in \Gamma$ we have*

$$i(\mu_{M\ddot{o}b}, \langle\gamma\rangle) = \ell(\gamma).$$

Proof. Let γ_-, γ_+ be the repulsive and attractive fixed points of γ. Let a be any point in $\partial X - \{\gamma_-, \gamma_+\}$. Let $I = [x, \gamma x]$ be an interval on the translation axis A_γ of γ of length $\ell(\gamma)$.

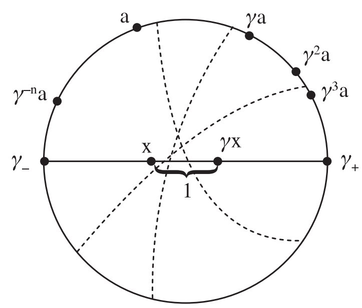

Figure 7. Intersection numbers of the Möbius current

Denote by $[\gamma_-, \gamma_+]$ the unique interval in ∂X with endpoints γ_-, γ_+, not containing a. For any integers n, m, $[\gamma^n a, \gamma^m a]$ denotes the subarc of ∂X between $\gamma^n a, \gamma^m a$ not meeting $[\gamma_-, \gamma_+]$ (see Figure 7). For every subsets J, K in $X \cup \partial X$, let $\mathcal{G}(J, K)$ denote the set of geodesics in $G(X)$ passing through both J, K.

The following lemma relates the Möbius current of $\mathcal{G}(I)$ (a Borel subset defined "internally") to the Möbius current of a Borel subset of $G(X)$ defined purely in terms of the boundary.

Lemma 4.8. *Let μ be a geodesic current giving measure* 0 *to the Borel sets of geodesics passing through a given point of $X \cup \partial X$. Then*

$$\mu(\mathcal{G}(I)) = \mu(\mathcal{G}([a, \gamma a], [\gamma_-, \gamma_+])).$$

Proof. Since any geodesic passing through I has to have one endpoint at a unique fundamental domain for the action of γ on the upper halfcircle (see Figure 7), we have

$$\mu(\mathcal{G}(I)) = \sum_{n \in \mathbb{Z}} \mu(\mathcal{G}([\gamma^n a, \gamma^{n+1} a[, [x, \gamma x])),$$

Since any geodesic with one endpoint in $[a, \gamma a]$ and the other in $[\gamma_-, \gamma_+]$ must pass through a unique fundamental domain for the action of γ on A_γ, we also have that

$$\mu(\mathcal{G}([a, \gamma a], [\gamma_-, \gamma_+])) = \sum_{n\in\mathbb{Z}} \mu(\mathcal{G}([a, \gamma a], [\gamma^{-n}x, \gamma^{-n+1}x[))$$

By invariance and Lemma 4.6, the n-th terms in the last two equations are equal. This ends the proof of Lemma 4.8. □

To finish the proof of the proposition, we combine the definition of the intersection number, Lemma 4.8, the symmetries of the crossratios and Corollary A.2 (for the last equality in the following sequence of equalities), to obtain

$$i(\mu_{M\ddot{o}b}, \langle\gamma\rangle) = \mu_{M\ddot{o}b}(\mathcal{G}(I)) = \mu_{M\ddot{o}b}(\mathcal{G}([a, \gamma a], [\gamma_-, \gamma_+]))$$

$$= |[a, \gamma a, \gamma_-, \gamma_+]| = |[\gamma_-, \gamma_+, a, \gamma a]| = \ell(\gamma).$$

□

We end this subsection by proving the implications (2) ⇒ (3) and (3) ⇒ (2) of Theorem B, inspired by [Ota1,Ota2].

Proposition 4.9. *Let X_1, X_2 be CAT(-1) spaces. Let Γ_1, Γ_2 two discrete subgroups of isometries of X_1, X_2 respectively, all of whose elements except the identity are hyperbolic isometries. Let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphism and $f : \partial X_1 \to \partial X_2$ be an equivariant homeomorphism.*

If f is Möbius, then $X_1/\Gamma_1, X_2/\Gamma_2$ have the same marked length spectrum.

Proof. Let $\gamma \in \Gamma_1$. Since f is an equivariant homeomorphism, it maps the attractive and repelling fixed points of γ to the attractive and repelling fixed points of $\rho(\gamma)$. Since f preserves the cross-ratio, by applying Corollary A.2, we have

$$\begin{aligned}\ell_1(\gamma) &= [\gamma_-, \gamma_+, a, \gamma a] = [f(\gamma_-), f(\gamma_+), f(a), f(\gamma a)]\\ &= [f(\gamma_-), f(\gamma_+), f(a), \rho(\gamma)f(a)] = \ell_2(\rho(\gamma)).\end{aligned}$$

□

Proposition 4.10. *Let M_1, M_2 be compact, connected, locally CAT(-1) spaces, with an isomorphism between their fundamental group. Assume that the boundaries of their universal covers X_1, X_2 are circles. If M_1, M_2 have the same marked length spectrum, then the boundary map is Möbius.*

Proof. If M_1, M_2 have the same marked length spectrum, then by Corollary 3.2, Proposition 4.7 and Theorem 4.3, the boundary maps sends the Möbius current of M_1 precisely to the Möbius current of M_2. By the definition of the Möbius current, this implies that the boundary map is Möbius. □

4.2. The Liouville current for negatively curved cone surfaces

The aim of this subsection is to modify the definition of the Liouville current for closed negatively curved surfaces to allow cone singularities.

Let X be a simply connected negatively curved cone surface, and $G_0(X) \subset G(X)$ the subset of nonsingular geodesics. Let k be an oriented geodesic segment in X with length $\ell(k) > 0$, parametrized by arclength. Consider the set $\mathcal{G}_0(k)$ consisting of all the nonsingular geodesics that meet k transversally (not in the endpoints of k). We define a set of coordinates on $\mathcal{G}_0(k)$. For any geodesic l in $\mathcal{G}_0(k)$, let t be the distance from the origin on k to the unique intersection point with k. Let θ be the angle of rotation between k and the geodesic l. Therefore $\mathcal{G}_0(k)$ can be identified with a subset of $[0, \ell(k)] \times [0, \pi]$ by $\Psi_k : l \mapsto (t, \theta)$. Letting k vary over all geodesic segments in X, we get an open cover of $G_0(X)$. Consider on $[0, \ell(k)] \times [0, \pi]$ the following measure

$$d\lambda = \frac{1}{2} \sin\theta \, d\theta \, dt.$$

Note that for any $x \in X \cup \partial X$, the measure for $d\lambda$ of the image in $[0, \ell(k)] \times [0, \pi]$ of the set of geodesics in $\mathcal{G}_0(k)$ passing through x is zero: the map f, which associates to t the unique θ such that the associated geodesic l goes through x, is Lipschitz where defined, hence its graph has λ-measure zero.

Also note that for this measure, the image of Ψ_k has full measure in $[0, \ell(k)] \times [0, \pi]$. There are only finitely many singularities on k. For any non singular $x \in k$, there are only countably many angles θ such that the geodesic l intersecting k at x with angle θ is singular.

Since Ψ_k is injective, the pull-back measure $d\sigma_k$ of $d\lambda$ on $\mathcal{G}_0(k)$ by Ψ_k is well defined.

Proposition 4.11. *The local measures $\{\sigma_k\}$ on $\mathcal{G}_0(k)$ match up to define a global measure on $G_0(X)$. Extending it, by giving measure 0 to the set $G(X) - G_0(X)$ of singular geodesic, yields a σ-finite regular Borel measure on $G(X)$.*

Proof. Let k and k' be two geodesic segments in X. We want to prove that $\sigma_k = \sigma_{k'}$ on $\mathcal{G}_0(k) \cap \mathcal{G}_0(k')$. Since the measure of the set of geodesics passing through a point is 0 for both measures, by a (countable) cutting process, we only have to prove the claim when there are no singularities in the disc having boundary made of k, k' and geodesic arcs connecting their endpoints. Then the proof for negatively curved Riemannian manifolds without singular points holds [San]. $\square$

If X covers a closed negatively curved cone surface S, the measure defined above on $G(X)$ is clearly invariant by the covering group, hence will be called the *Liouville current* of S, and will be denoted by μ_{Liou}. (We will also denote by μ_{Liou} the measure on $\partial^2 X$ defined similarly.) Note that the following fact immediately holds by integration:

Proposition 4.12. *For any geodesic segment k in X we have*

$$\mu_{Liou}(\mathcal{G}(k)) = \mu_{Liou}(\mathcal{G}_0(k)) = \ell(k). \qquad \square$$

The following proposition relates the Möbius and the Liouville currents.

Theorem 4.13. *Let S be a closed negatively curved cone surface, then*

$$\mu_{M\ddot{o}b} = \mu_{Liou}.$$

Proof. By Propositions 4.7 and 4.12, both currents have the same intersection numbers. So the result follows from Theorem 4.3. $\square$

The main application of the above proposition is that we now know that the Möbius measure of the set of geodesics passing through any geodesic segment of X equals the length of the segment. This will be essential to show that the boundary map, if Möbius, is an extension of an isometry.

4.3. The Lebesgue measure for the geodesic flow on a negatively curved cone surface

Let S be a negatively curved cone surface, P its set of singularities. Let $T^1(S)$ be the Borel set of tangent vectors $v \in T^1(S-P)$ such that no geodesic directed by v is singular. Note that the geodesic flow ϕ^t is well defined on $T^1(S)$. (As usual, $\phi^t(v)$ is the tangent vector to the unique geodesic defined by v at the unique point at distance t from the base point of v.)

Even if $S-P$ is non complete, we have on $T^1(S-P)$ well defined measure, the Lebesgue measure, invariant by the geodesic flow where defined. In canonical local charts on the unit tangent bundle, it is the product of the volume form of the manifold by the Lebesgue measure on the unit sphere. See for instance [KH, page 205], where this measure is called the Liouville *measure*. To avoid confusion with the Liouville *current*, we prefer to call it the Lebesgue measure. We will denote the restriction of the Lebesgue measure to $T^1(S)$ by μ_{Leb}, and still call it the *Lebesgue measure*. This measure is invariant by the geodesic flow ϕ^t.

Since there are only countably many directions of singular geodesics at a given point, the subset $T^1(S)$ has full Lebesgue measure in $T^1(S-P)$.

There is a continuous action of $\mathbb{S}^1$ on $T^1(S)$, $v \to \theta \cdot v$, which is defined for a given v except on a countable subset of $\mathbb{S}^1$, hence whose domain has full measure in $T^1(S) \times \mathbb{S}^1$ for the product of the Lebesgue measures.

Let X be the universal cover of S, which is a negatively curved cone surface (with singularities at the lifts of P). Any v in $T^1(X)$ defines a unique geodesic in X, which will be denoted by l_v. One has a well defined map $f_1 : T^1(X) \to \partial^2 X - \Delta$, which associates to each v the points at infinity of l_v. It is clear that

f_1 is measurable and the fibers of f_1 are the geodesics in X. Note that, unlike the nonsingular case, f_1 *is not continuous.* Fixing a base point in X, one get a measurable bijection $f : T^1(X) \to (\partial^2 X - \Delta) \times \mathbb{R}$, whose first component is f_1, the second being the signed distance from the projection of the base point to the origin of v.

Lemma 4.14. *The pushforward of the Lebesgue measure by f satisfies:*

$$df_* \mu_{Leb} = d\mu_{Liou}\, ds,$$

where s denotes the arclength along geodesics.

Proof. On the set corresponding to singular geodesics, both measures have measure 0. Outside it, the usual proof for negatively curved manifolds applies. □

4.4. Negatively curved surfaces with same marked length spectrum

The aim of this final part is to prove that (2)+(3) implies (4) in Theorem C. We will follow closely J.-P. Otal's proof in [Ota1], emphasizing mainly the points where the proofs are different.

Let S be a closed connected surface. Let m, m' be two negatively curved cone metrics on S, having the same marked length spectrum. We will add the subscript m, m' to tell which metric we are considering. For instance, $X_m, X_{m'}$ will be the universal covers of S with the metrics lifted from m, m'. The hypotheses (2)+(3) imply that we have an equivariant homeomorphism $\tilde{\phi} : \partial X_m \to \partial X_{m'}$ such that $\tilde{\phi} \times \tilde{\phi}$ preserves the Möbius currents.

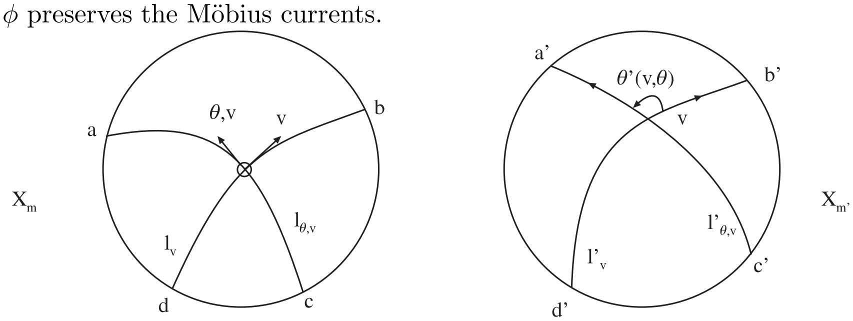

Figure 8. Angle correspondance

For almost every (v, θ) in $T^1(X_m) \times [0, \pi]$, we will define an angle $\theta'(v, \theta) \in [0, \pi]$. For $\theta = 0, \pi$, set $\theta'(v, \theta) = 0, \pi$ respectively.

Assume that $\theta \neq 0, \pi$ and that $\theta \cdot v$ is a non singular direction. (Note that the set of (v, θ) such that $\theta \cdot v$ is a singular direction has measure 0 in $T^1(X_m) \times [0, \pi]$.) Let $l'_v, l'_{\theta \cdot v}$ be the geodesics in $X_{m'}$ whose endpoints are the images by $\tilde{\phi}$ of the endpoints of l_v and $l_{\theta\cdot v}$ (see Figure 8). The endpoints in ∂X_m of the geodesics l_v and $l_{\theta\cdot v}$ are intertwined. Since $\tilde\phi$ is an homeomorphism, so are the endpoints of l'_v and $l'_{\theta\cdot v}$. So these two geodesics have to intersect.

Lemma 4.15. *Under the above hypotheses, the geodesics l'_v and $l'_{\theta\cdot v}$ are non singular, hence meet in one and only one point.*

Proof. Assume by absurd that l'_v is singular. Let x' be a point on l'_v with cone angle $> 2\pi$. Let b', d' be the endpoints of l'_v. There exist closed intervals U', V' in $\partial X_{m'}$, containing b', d', with non empty interiors, with $U' \cap V' = \emptyset$, such that for any $a'' \in U'$ and $c'' \in V'$, the geodesic between a'', c'' goes through x'. Hence the crossratio $[a'', b', c'', d']$ vanishes for all $a'' \in U' - \{b'\}$ and $c'' \in V' - \{d'\}$.

Since ϕ^{-1} is Möbius, this contradicts the fact that l_v is nonsingular. □

We will denote by $\theta'(v, \theta)$ the angle (in $]0, \pi[$) between the geodesics l'_v and $l'_{\theta\cdot v}$ at the unique intersection point given by the previous lemma.

By equivariance, θ' induces a map defined a set of full measure of $T^1_m(S) \times [0, \pi]$ for the product of the Lebesgue measures, with values in $[0, \pi]$, which is clearly measurable. (A priori, this map is not continuous, contrary to the nonsingular case of J.-P. Otal. Indeed, the map, which associates to a unit tangent vector the endpoints of the geodesic it defines, is no longer continuous.)

Let us denote by $\mathcal{V}(T^1_m(S))$ the total volume for the Lebesgue measure $d\mu_{Leb}$ on $T^1_m(S)$.

Consider the following average of the angles $\theta'(v, \theta)$

$$\Theta^{'}(\theta) = \frac{1}{\mathcal{V}(T^1_m(S))} \int_{T^1_m(S)} \theta^{'}(v, \theta)\, d\mu_{Leb}.$$

Proposition 4.16, Proposition 4.17 and Lemma 4.18 are easy modifications of J.-P. Otal's results in [Ota1, Proposition 6,7 and Lemma 8]. We only emphasize the differences with the non singular case. For instance, the lack of continuity forces us to use "measurable" arguments.

Proposition 4.16. *The following properties hold for $\Theta^{'}$:*

1. $\Theta^{'} : [0, \pi] \to [0, \pi]$ *is increasing,*

2. $\Theta^{'}$ *commutes with the symmetry with respect to $\frac{\pi}{2}$:*

$$\forall \theta,\ \Theta^{'}(\pi - \theta) = \pi - \Theta^{'}(\theta),$$

3. $\Theta^{'}$ *is super-additive:*

$$\forall \theta_1, \theta_2 \text{ such that } \theta_1 + \theta_2 \in [0, \pi],\ \text{ we have } \Theta^{'}(\theta_1 + \theta_2) \geq \Theta^{'}(\theta_1) + \Theta^{'}(\theta_2).$$

*Proof.*The map $\Theta^{'}$ is measurable and satisfies $\Theta^{'}(0) = 0, \Theta^{'}(\pi) = \pi$. It is positive on $]0, \pi[$ as an average of a positive measurable function. In particular, the assertion 1) follows from the assertion 3). The rest of the proof follows from [Ota1, Prop. 6] where no continuity is needed. The main points used here are the almost everywhere invariance of the metric by rotation and the singular Gauss-Bonnet formula, which does hold for negatively curved cone surfaces. □

We introduce further notations before the next proposition. Let F be a real valued measurable convex function defined on the interval $[0,\pi]$. According to the Jensen inequality (cf. [Ru, page 63]), we have that for every $\theta \in [0,\pi]$:

$$F(\Theta^{'}(\theta)) \leq \frac{1}{\mathcal{V}(T^1_m(S))} \int_{T^1_m(S)} F(\theta^{'}(v,\theta))\, d\mu_{Leb}.$$

Moreover, if F is strictly convex, equality holds in the above integral if and only if the function $v \to \theta^{'}(v,\theta)$ is constant. We note that since $\theta^{'}$ is a bounded measurable function, we can integrate the second term in the above integral with respect to the measure $\sin\theta\, d\theta$ on the interval $[0,\pi]$. Applying Fubini we obtain:

$$\int_0^\pi F(\Theta^{'}(\theta)) \sin\theta\, d\theta \leq \frac{1}{\mathcal{V}(T^1_m(S))} \int_{T^1_m(S)} \left(\int_0^\pi F(\theta^{'}(v,\theta)) \sin\theta\, d\theta \right) d\mu_{Leb}.$$

Set $F^{'}(v) = \int_0^\pi F(\theta^{'}(v,\theta)) \sin\theta\, d\theta$. Thus, the second term in the above integral is the average of the measurable function $F^{'}$ on $T^1_m(S)$.

Proposition 4.17. [Ota1, Proposition 7] *For any convex function F as above we have*

$$\int_0^\pi F(\Theta^{'}(\theta)) \sin\theta\, d\theta \leq \int_0^\pi F(\theta) \sin\theta\, d\theta.$$

Proof. Let $\gamma \in \pi_1 S$, and $A_\gamma \subset X_m, A'_\gamma \subset X_{m'}$ be the translation axes, parametrized by arclength, oriented from the repulsive fixed point to the attractive one. As before (see in particular Lemma 4.15), the boundary map $\tilde{\phi}$ induces a measurable bijection between the subset of $A_\gamma \times]0,\pi[$ of nonsingular geodesics meeting transversaly A_γ in a point, to the corresponding subset of $A_\gamma \times]0,\pi[$. This map $(t,\theta) \mapsto (t',\theta')$ is equivariant under γ. It sends the measure $\sin\theta\, d\theta dt$ to the measure $\sin\theta'\, d\theta' dt'$, since the boundary map preserves the Liouville currents.

Note that the currents supported on the set of endpoints of all translation axes are dense (see [Bon3, Theorem 7]).

The proof now follows as in [Ota1, p. 159-160]. □

Lemma 4.18. *Let Υ be a measurable increasing function on $[0,\pi]$. Suppose that*
1. Υ is super-additive and commutes with the symmetry with respect to $\frac{\pi}{2}$.
2. For any convex function F defined on $[0,\pi]$, we have

$$\int_0^\pi F(\Upsilon(\theta)) \sin\theta\, d\theta \leq \int_0^\pi F(\theta) \sin\theta\, d\theta.$$

The Υ is the identity.

Proof. Since Θ is measurable and increasing, we can follow the same arguments as [Ota1, Lemma 8]. □

We are now ready to end the proof of Theorem C.

Proof of (2)+(3) implies (4) in Theorem C. The function $\Theta^{'}$ as defined above satisfies all the hypotheses of Lemma 4.18. Therefore we have $\Theta^{'} = \mathrm{Id}$. In particular, it is additive. Using the analysis of the equality case in the proof of Proposition 4.16 (3), we conclude that the image of three nonsingular geodesics intersecting at the same point in the domain surface is three geodesics intersecting in one and only one point.

Let $\phi : X_m \to X_{m'}$ be the map sending the intersection point of two non singular geodesic in X_m to the unique intersection point of the geodesics in $X_{m'}$ between the images by $\tilde{\phi}$ of the endpoints of the geodesics in X_m. By the above discussion, this map is well defined and equivariant.

Let $p, q \in (S, m)$ and let $\phi(p), \phi(q)$ be their images in (S, m'). By Theorem 4.13 and Proposition 4.12, we have

$$d_m(p, q) = \mu_{M\ddot{o}b}(\mathcal{G}_m([p, q]))$$

since $\tilde{\phi}$ is an homeomorphism, the image by $\tilde{\phi} \times \tilde{\phi}$ of $\mathcal{G}_m([p, q])$ is $\mathcal{G}_{m'}([\phi(p), \phi(q)])$. Since $\tilde{\phi}$ preserves the Möbius currents, we have

$$d_m(p, q) = d_{m'}(\phi(p), \phi(q)).$$

Since ϕ is an isometry outside the singularities it must be an isometry everywhere. This ends the proof of Theorem C. □

Remark. We note that this proof also holds for some other singular negatively curved metrics (which are "almost everywhere" Riemannian, in some precise sense to be defined).

Appendix. New metrics on the punctured boundary

Assume first that X is the real hyperbolic n-space. Let $a \in \partial X$, and consider the upper half space model where a is at infinity. One has on $\partial X - \{a\}$, which is now $\mathbb{R}^{n-1}$, a metric, the Euclidian metric. This metric is invariant, up to homotheties only, by the isometries of X fixing a. That is, the upper half space model is canonically defined by a choice of point at infinity a and by a choice of an horosphere centered at a.

Let now X be any proper CAT(-1) space. In this appendix, we will define a family of metrics $\{d_{a,\mathcal{H}}\}$ on $\partial X - \{a\}$, for $a \in \partial X$ and $\mathcal{H}$ an horosphere centered at a, such that an isometry of X fixing a acts by homotheties of this metric. This makes more precise the construction of [GH] section 8 for general hyperbolic spaces (in Gromov's sense).

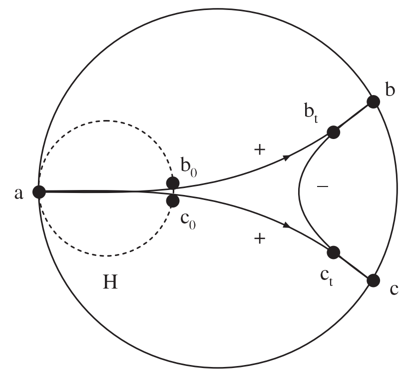

Figure 9. Euclid-Cygan metric on the boundary

For b, c in $\partial X - \{a\}$, define

$$d_{a,\mathcal{H}}(b,c) = \lim_{t \to +\infty} e^{-\frac{1}{2}(2t - d(b_t, c_t))}$$

where $t \mapsto b_t, c_t$ are the geodesics in X (parametrized by arclength) respectively from a to b and from a to c, such that b_0, c_0 are on $\mathcal{H}$. This limit exists by a Cauchy argument because inside some small enough neighborhood of b (resp. c), the geodesic between b_t, c_t lies arbitrarily close to the geodesic between a, b (resp. a, c). Let $r : [0, +\infty[\to X$ be a geodesic ray whose point at infinity is a (so that $r(t)$ tends to a as $t \to +\infty$), and with $r(0)$ belonging to $\mathcal{H}$. Then it is easy to prove that:

$$d_{a,\mathcal{H}}(b,c) = \lim_{t \to +\infty} e^{-t} d_{r(t)}(b,c).$$

Now $d_{a,\mathcal{H}}$ is clearly a distance. For instance, the triangle inequality follows from the one for the visual distances and the previous formula.

It is also easy to check that on $\partial X - \{a\}$, the distance $d_{a,\mathcal{H}}$ induces the usual topology, and that the conformal structure defined by $d_{a,\mathcal{H}}$ is the same as the one defined by any d_x for $x \in X$:

$$d_{a,\mathcal{H}}(b,c) = e^{-\frac{1}{2}(B_b(x, h_{ab}) + B_c(x, h_{ac}))} d_x(b,c)$$

where h_{ad} is the intersection point of the horosphere $\mathcal{H}$ with the geodesic between

a and some point d in ∂X. So that the following limit does exist:

$$\lim_{c \to b} \frac{d_{a,\mathcal{H}}(b,c)}{d_x(b,c)} = e^{-B_b(x,h_{ab})}.$$

For any isometry g of X, one clearly has

$$d_{ga,g\mathcal{H}}(gb,gc) = d_{a,\mathcal{H}}(b,c).$$

In particular, the distance $d_{a,\mathcal{H}}$ is unchanged by all isometries preserving a and $\mathcal{H}$.

Before proving other properties of these metrics, let us make the following definition.

Let $\mathcal{H}, \mathcal{H}'$ be two horospheres centered at the same point a. For any geodesic ℓ with endpoint a, let x, y be the intersection points of ℓ with $\mathcal{H}$ and $\mathcal{H}'$ respectively. By the cocycle relation, $B_a(x,y)$ does not depend on the geodesic ℓ, and will be denoted by $\sigma(\mathcal{H},\mathcal{H}')$. Note that $\sigma(\mathcal{H},\mathcal{H}') = -\sigma(\mathcal{H}',\mathcal{H})$ and $|\sigma(\mathcal{H},\mathcal{H}')| = d(x,y)$ for the above x, y. If an isometry g fixes a, define

$$j_{\mathcal{H}}g(a) = e^{\sigma(\mathcal{H},g\mathcal{H})}.$$

Note that there is no incompatibility in the notations, since $j_{\mathcal{H}}g(a) = j_x g(a)$ for any x in $\mathcal{H}$.

Now, if an isometry g fixes a, then

$$d_{a,\mathcal{H}}(gb,gc) = j_{\mathcal{H}}g(a)\, d_{a,\mathcal{H}}(b,c)\,. \tag{10}$$

The following formula generalizes the value of the crossratio of four points on the sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ when one point is ∞.

Lemma A.1. *For any distinct four points a, b, c, d on the boundary, and any horosphere $\mathcal{H}$ centered at a, one has*

$$e^{[a,b,c,d]} = \frac{d_{a,\mathcal{H}}(b,d)}{d_{a,\mathcal{H}}(b,c)}.$$

Proof. This formula is easily obtained by letting x tends to a in Equation (7). $\square$

Using the symmetries of the crossratios, one easily gets other formulas. Note that the right handside is thus independant of $\mathcal{H}$.

Corollary A.2. *Let γ be an hyperbolic isometry of X, with γ_-, γ_+ respectively the* *repulsive and attractive fixed point in ∂X. Let a by any point in $\partial X - \{\gamma_-, \gamma_+\}$. Then*

$$[\gamma_-, \gamma_+, a, \gamma a] = \ell(\gamma).$$

Proof. Since γ_-, γ_+ are fixed by γ, according to Lemma A.1 and equation (10), one has

$$e^{[\gamma_-, \gamma_+, a, \gamma a]} = j_{\mathcal{H}} g(g_-)^{-1}.$$

Since g_- is repulsive, any horoshere $\mathcal{H}$ is mapped by γ to an horosphere $\gamma\mathcal{H}$ whose horoball contains $\mathcal{H}$ in its interior. If x is the intersection point of the translation axis of γ with $\mathcal{H}$, then $j_{\mathcal{H}} g(g_-)^{-1} = e^{d(x, \gamma x)}$. The result follows. □

Corollary A.3. *Let X_1, X_2 be CAT(-1) spaces, and $A \subset \partial X_1$. If a map $f : A \to \partial X_2$ is Möbius, then it is an homothety for the above metrics, in the sense that $d_{f(a),\mathcal{H}'}(f(b), f(c)) = \lambda_{a,\mathcal{H},\mathcal{H}'} d_{a,\mathcal{H}}(b, c)$.*

Proof. If f preserves the crossratios, by Lemma A.1, one has

$$\frac{d_{f(a),\mathcal{H}'}(f(b), f(c))}{d_{a,\mathcal{H}}(b, c)} = \frac{d_{f(a),\mathcal{H}'}(f(b), f(d))}{d_{a,\mathcal{H}}(b, d)}$$

for all b, c, d and since the right handside is independant of c, by symmetry of the distance, the result follows. □

Acknowledgements

M. Culler and P.-B. Shalen introduced the first author to the Patterson-Sullivan measure. We thank them both for an enormous amount of conversations on the subject.

The research on this paper started when the second author was visiting the department of mathematics in Cornell University during the fall semester of 1994. It came to its end while the second author was visiting the ENS Lyon on a "Poste rose" CNRS during the summer of 1995. We thank both institutions for their hospitality and support. We thank E. Ghys for giving us C. Yue's papers, F. Haglund for his examples, S. Barré for his proof of Lemma 2.10 in the affine case, and J.-P. Otal for his encouragment and crucial help leading to the proof of Theorem C.

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(Received: December 19, 1995)