Singular level curves of harmonic functions, conformal mappings and emerging applications to shape recognition of planar-domains

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Abstract— *In spite of significant advances in computer graphics and computer vision for description and recognition of rigid shapes and objects, the problem of description and recognition of non-rigid shapes and objects is still open (see for instance [12]). In earlier studies [8], [9], [10] and [3], we proposed an approach, based on complex and harmonic analysis, aiming to describe and analyze non-rigid shapes. In this note, we describe a step in our program to supply a* canonical *set of shapes, very much like a dictionary with mathematical structures, that intrinsicially determine the shape and its classification. Our goal is to provide a computational method of representing 2D shapes where every planar shape will be assigned a unique fingerprint, which is a conformal map of the shape to a canonical shape in the plane or in space. We believe that the study presented in this paper can be extended to describe and recognize both rigid and non-rigid shapes and objects. The main focus of this article is the case of planar domains of high genus since the case of simply-connected and doubly-connected domain was treated in the references mentioned above.*

Keywords: Domain decomposition, Image processing, Conformal mappings

1. Introduction

We are concerned with research problems that belong to the field of Shape Recognition, which is an important and active topic within the scope of Computational Science. For instance, The study of 2D shapes and their similarities is of a central importance in the field of computer vision. It includes the classification and recognition of various kinds of shapes that appear in pictures and videos, defining natural distances between 2D shapes, and creating a *canonical set* of shapes, very much like a dictionary with mathematical structures that are inherently relevant to the classification task.

Conformal maps have had central applications in Pure Mathematics. These maps preserve the angles between any two intersecting curves in the plane. However, the *existence* theory of such maps depends on delicate theorems from the field of Partial Differential Equations in Pure Mathematics, and the computational aspects of their approximation remains an area of active research.

In the 19th and the beginning of the 20th century, the recognition of Riemann surfaces and the conformal structures defined on them was an important milestone in the development of Mathematics. This motivated the formulation of the *uniformization theorem*.

The uniformization theorem [2] implies that any simply connected Riemann surface is *conformally equivalent* to one of three known Riemann surfaces: the open unit disk, the complex plane or the Riemann sphere. This remarkable theorem is a vast generalization of the celebrated *Riemann mapping theorem* asserting that a non-empty simply connected open subset of the complex plane (which is not the whole of it), is conformally equivalent to the open unit disk in the complex plane.

Conformally equivalent means that there exists an angle-preserving map from the given set (the domain surface) to the open disk (other model surfaces) which is one-to-one and onto, and that map has an inverse with the same properties. Such a map preserves the shape of any sufficiently small figure, while possibly rotating and scaling (but not reflecting) it.

These theorems are important in other fields such as Physics and Computer Science. Usually, the input is a 2-dimensional surface, such as the surface of a ball with many corners, or a donut (with one hole or more). These objects often show up in applications, and Uniformization allows one (in principle) to put a set of global coordinates on the input surfaces, thereby allowing useful computations.

In this note, by introducing *singular level curves discrete harmonic functions*, we advance our results in [8], [9], [10], [3] towards our goal of representing 2D shapes: every planar shape will be assigned a unique *fingerprint*, which is a *conformal map* of the shape to a *canonical* shape in the plane or in space, where a *finite* and computable set of parameters determine if the shapes are conformally equivalent or not.

2. Domain decomposition along singular level curves of harmonic functions

There is a classical theory of conformal uniformization for domains in the Riemann sphere that are not simply connected. Keeping our previous notation, let Ω be a domain in the Riemann sphere. Koebe proved that Ω is conformally homeomorphic to some domain Ω^* whose boundary components are circles (or points). Furthermore, Ω^* is unique up to Möbius transformations. Such a domain is called a *circle domain*. This is an existence theorem and a concrete description of such a conformal map is desirable.

It is reasonable to first construct a *discrete uniformization map*, that is a rough approximation to the desired conformal map. This approximation is quite hard to achieve since the input is usually a coarse type of information (such as a triangulation of the domain), and the output should consist of a map from the domain to a surface with some geometric structure. This map should have nice properties (see for instance [8], [9]). Once this step is complete, one then tries to prove convergence of maps and convergence of the output objects attained to concrete geometric objects, under suitable conditions.

In [10, Theorem 4.7], such a scheme in the case of doubly-connected domains (i.e., when the input is a planar annulus) was proved. Specifically, it was shown that given *any* planar annulus, say with a polygonal boundary, there exists a scheme of discrete approximation which yields the best possible approximation (i.e., in the L_∞ norm) to the conformal map of this annulus to a round one.

The scheme is based on constructing a sequence of pairs of discrete harmonic functions where each pair is defined on a triangulation of the given annulus. As the triangulations get finer and finer, it is shown in [10] that the limiting pair converges to the real and imaginary parts, respectively, of the conformal map. The reader is invited to view [10] and [3] for the details.

With the above structure in place, a natural extension that comes up is: What are the analogous results when the domain under consideration has two holes? One hundred holes? One should note that these questions are related to applications. For example, a face can be roughly drawn as a non-planar domain with several holes (eyes, mouth, and nostrils).

To address this situation we are setting forth the mathematical and computational foundations. One novel idea is to decompose the given domain into simpler regions that can be assembled together to create a uniform model. However, as the case of one hole (i.e., annulus) shows, there is an increasing level of difficulty as the number of holes is increased. For instance, it turns out that the canonical images (even in the case of domains with two holes) will no longer be planar.

Indeed, in [8, Section 2] is was shown that even if the planar domain has many holes, there is a procedure to decompose it into simpler regions by cutting along singular level curves of certain discrete harmonic functions, where each region is either an annulus or very close to being an annulus. The procedure is based on successively cutting the region along singular level curves of a discrete harmonic function that approximates the real part of the conformal mapping of the domain. The following figures show two polygonal annuli and the approximation for their conformal images.

3. Algorithmic and Computational approach for a decomposition of a domain along singular level curves

First, we introduce two important definitions: A *region object* needs to contain a subset of $\mathbb{R}^2$ by giving the piecewise-linear outer boundary and two piecewise-linear inner boundaries giving the boundaries of the two holes in the subset.

A *triangulation object* needs to contain the coordinates of the vertices in a triangulation, the indices of the edges drawn between these vertices, and the triplets of indices of the triangles in the triangulation. In addition, another data structure we will call the *triangulation topology* needs to be calculated and included as well, which stores which triangles are adjacent to other triangles along an edge.

The algorithm works as follows: first we need to compute the triangulation object from the region object. This is done using the function `triangulate_acute(R)`. We compute an *acute triangulation* (a triangulation where every triangle is acute) of the region, which is done using the aCute software. This gives an initial acute triangulation of the region, but to construct a sequence of acute triangulations with diminishing mesh, we need to also have a way of refining this triangulation. To preserve the angles of the triangulation, which will then preserve the acuteness of the triangulation, we use what we will call the Hojo refinement [11]. To apply this refinement to a single triangle, take the midpoint of each edge and connect them to form a new triangle, which then partitions the original triangle into four sub-triangles with the same angles as the original.

We now solve the Dirichlet problem on the region using our acute triangulation. This is mostly based on the Mathematica function `NDSolveValue` [13] which is well documented. We let `f` denote the function which solves this PDE, which is 0 on the inner boundaries and 1 on the outer boundary.

To compute the singular level curve, we first need to find the vertex which lies in the singular level curve by extensively using the index formula (5.1) in [8] which is taken from [4]. For every vertex v in the triangulation T, we use the edge information to compute the neighbors of v so that the number of sign changes between those neighbors can be computed as well. This is denoted $\mathrm{Sgc}_f(v)$, and with this computed we can compute the index of the vertex v using the formula:

$$\mathrm{Ind}_f(v) = 1 - \frac{\mathrm{Sgc}_f(v)}{2}.$$

Compute this index for every vertex, and the singular vertex can be identified by having an index different than zero. With

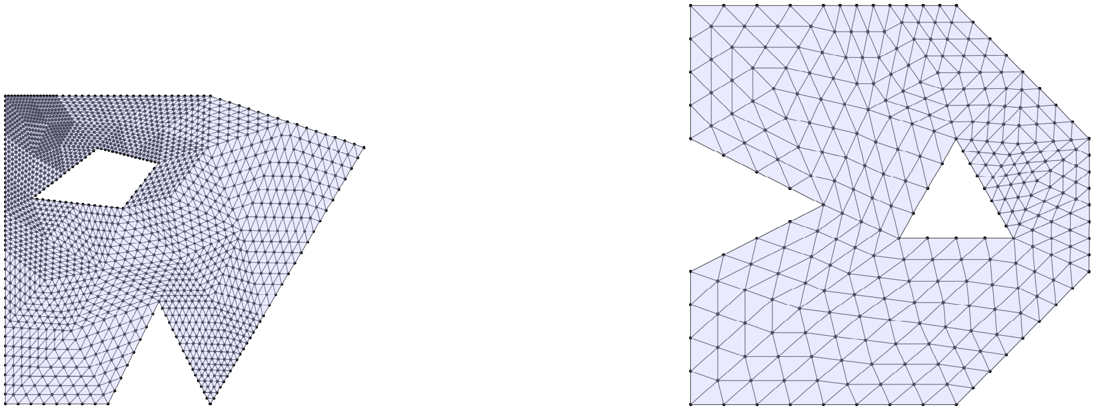

Fig. 1: The triangulated polygonal annuli

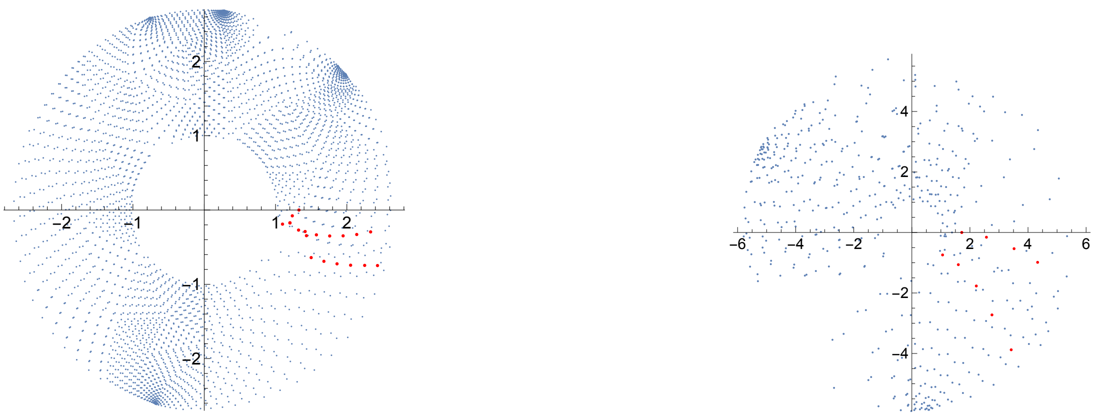

Fig. 2: The images of the polygonal annuli under the conformal map

the singular vertex, say v_s in hand, we can compute $f(v_s)$ to compute the height of the singular level curve, and then the singular level curve can easily be computed by finding the piecewise-linear curve at that height (remembering that the function f is linear on each triangle in the triangulation).

4. The algorithm

Let us provide a short description of the algorithm used to compute and produce the examples in this article. Our algorithm relies on the aCute software to generate acute triangulations, which are defined as triangulations where every triangle is acute. References for this package can be found in the bibliography, and the contact author for the software is Alper Üngör.

5. Numerical Results

In the next page, several figures depict the way we decompose such domains into simper regions via the *singular* level curves of the discrete harmonic functions we discussed in previous sections. The work of assembling the resulting regions (each one is an annulus) into a canonical family of domains is under our current research.

Algorithm 1: Algorithm to Compute the Singular Level Curve

Input: A region object `R` with genus two
Output: The singular level curve of the region object `R`

```
T = triangulate_acute(R)
f = solve_pdf(T)
singular_vertex = find_singular_vertex(T)
singular_height = f(singular_vertex)
singular_level_curve = compute_level_curve(T, singular_height)
```

Fig. 3: Planar domain with two holes

Fig. 4: The triangulations of the domains

Fig. 5: The singular curves of the harmonic function

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