Counting horoballs and rational geodesics

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Abstract

Let M be a geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. The asymptotics of the number of geodesics in M starting from and returning to a given cusp, and of the number of horoballs at parabolic fixed points in the universal cover of M, are studied. The case of $\mathrm{SL}(2,\mathbb{Z})$ and of Bianchi groups is developped. [1]

1 Introduction

Let M be a non elementary geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. A geodesic line starting from a given cusp e is *rational* if it converges to e, and *irrational* if it accumulates inside M. Motivated by problems arising from diophantine approximation, we developed in [6] a theory of approximation of irrational geodesics by rational ones.

As introduced in [6], the *depth* $D(r)$ of a rational line r is the length of the subsegment of r between the first and last meeting point with the boundary of the maximal Margulis neighborhood of the cusp e. We proved in [6] that the set of depths of rational lines is a discrete subset of $\mathbb{R}$ with finite multiplicities. So we may define the *depth counting function* $\mathcal{N}_e : \mathbb{R} \to \mathbb{N}$, with $\mathcal{N}_e(x)$ the number of rational geodesics whose depth is less than x.

Let $\widetilde{M}$ be a fixed universal cover of M, with covering group Γ, and let x_0 be a base point in $\widetilde{M}$. Recall that (see for example [1]) the *Poincaré series* of any discrete group G of isometries of $\widetilde{M}$ is

$$P(s) = \sum_{g \in G} e^{-s\, d(x_0,\, gx_0)}$$

for any s in $\mathbb{R}$. This series converges if $s > \delta_G$ and diverges if $s < \delta_G$ for some δ_G which is independent of x_0. Moreover, $0 < \delta_G < +\infty$ and δ_G is called the *critical exponent* of G. We say that G is *divergent* if $P(\delta_G)$ diverges.

Choose a parabolic fixed point ξ_0 on the boundary $\partial\widetilde{M}$ of $\widetilde{M}$, corresponding to e. Let Γ_0 be its stabilizer. Recall that if $\widetilde{M}$ is a rank 1 symmetric space (of non compact type), or if Γ_0 is divergent, then $\delta_G > \delta_{\Gamma_0}$ (see [4], where the authors also give an interesting example where equality holds).

Theorem 1.1 *If* $\delta_\Gamma > \delta_{\Gamma_0}$, *then* $\limsup_{x\to+\infty} \dfrac{\log \mathcal{N}_e(x)}{x} = \delta_\Gamma$.

[1] **AMS codes:** 53 C 22, 11 J 06, 30 F 40, 11 J 70. **Keywords:** rational geodesic, negative curvature, cusp, horoball.

This result on the asymptotic growth of the depths of rational lines is easily deduced from the following statement. We define the *relative Poincaré series* of (Γ, Γ_0) as

$$P_0(s) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0} e^{-s\, d(H_0, \gamma H_0)}$$

for any s in $\mathbb{R}$, where H_0 is any fixed horosphere centered at ξ_0. It is easy to see that $d(H_0, \gamma H_0)$ depends only on the double coset of γ, and that the convergence or divergence of the relative Poincaré series does not depend on the choice of H_0.

Theorem 1.2 *If $\delta_\Gamma > \delta_{\Gamma_0}$, then $P_0(s)$ converges if and only if $P(s)$ converges.*

These results apply in particular to the arithmetic hyperbolic manifolds or orbifolds. For example, let $\mathcal{O}$ be the ring of integers of a number field K, having finite group of units O^* (i.e. K is $\mathbb{Q}$ or an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$). Let N be the norm on K, i.e. $N(x) = x$ if $K = \mathbb{Q}$ and $N(x) = |x|^2$ if $K = \mathbb{Q}(\sqrt{-d})$. Applying Theorem 1.1 to the modular orbifold $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$ if $K = \mathbb{Q}$ or to the Bianchi orbifold $\mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O})$ if $K = \mathbb{Q}(\sqrt{-d})$, one gets

Corollary 1.3 *Let $\varphi_{\mathcal{O}}(x)$ be the cardinal of the set of $\frac{p}{q}$ mod $\mathcal{O}$, where p, q belongs to $\mathcal{O}$ with $(p,q) = 1$ and $0 < N(q) \leq x$. Then*

$$\limsup_{x \to +\infty} \frac{\log \varphi_{\mathcal{O}}(x)}{\log x} = 2.$$

This result is well-known for $K = \mathbb{Q}$, where $\phi_{\mathbb{Z}}(x) = \sum_{k=0}^{x} \phi(k)$ and ϕ is the Euler fonction. A much more precise result than Corollary 1.3 is given in section 4. Using the techniques of section 4 when $r_1 + r_2 = 1$, one can hope for an analogous result for other number fields, in connection with the counting of horoballs in $(\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}$ under the irreducible arithmetic lattice $SL_2(\mathcal{O})$ having $\mathbb{Q}$-rank 1 and $\mathbb{R}$-rank $r_1 + r_2$, where r_1 (resp. $2r_2$) is the number of real (resp. complex) embeddings of K. This will be developed later.

In constant curvature, some of our results are weaker than those in [3], which was brought to our attention only after the first version of this paper had been written.

2 Definitions

We will use the notations and definitions of [6], that we recall here briefly for the sake of completeness. We refer to [6] for proofs and comments on these notions.

Let M be a (smooth) complete Riemannian n-manifold with pinched negative sectional curvature $-b^2 \leq K \leq -a^2 < 0$. Fix a universal cover $\widetilde{M}$ of M, with covering group Γ.

The boundary $\partial\widetilde{M}$ of $\widetilde{M}$ is the set of asymptotic classes of geodesic rays in $\widetilde{M}$. The space $\widetilde{M} \cup \partial\widetilde{M}$ is endowed with the cone topology. The *limit set* $\Lambda(\Gamma)$ is the set $\overline{\Gamma x} \cap \partial\widetilde{M}$, for any x in $\widetilde{M}$. Let $C\Lambda(\Gamma)$ be the convex hull of the limit set of Γ.

A point ξ in $\Lambda(\Gamma)$ is a *conical limit point* of Γ if it is the endpoint of a geodesic ray in $\widetilde{M}$ which projects to a geodesic in M that is recurrent in some compact subset. A point ξ in $\Lambda(\Gamma)$ is a *bounded parabolic point* if it is fixed by some parabolic element in Γ, and if the quotient $(\Lambda(\Gamma) - \{\xi\})/\Gamma_\xi$ is compact, where Γ_ξ is the stabilizer of ξ.

We assume that the group Γ is *geometrically finite*, i.e. that every limit point of Γ is conical or bounded parabolic (see [2] for more details). We also assume that Γ is *non elementary*, i.e. that its limit set contains at least 3 points.

Assume that M has at least one *cusp* e, i.e. an asymptotic class of minimizing geodesic rays in M along which the injectivity radius goes to 0. We say that a geodesic ray *converges* to e if some subray belongs to the class e.

Choose a parabolic fixed point ξ_0 on the boundary $\partial\widetilde{M}$ of $\widetilde{M}$, which is the endpoint of a lift of a geodesic ray converging to e. Let Γ_0 be its stabilizer in Γ. Let H_0 be the horosphere centered at ξ_0 such that the horoball HB_0 bounded by H_0 is the maximal horoball centered at ξ_0 such that the quotient of its interior by Γ_0 embeds in M under the canonical map $\widetilde{M} \to M$. The subset $\mathrm{int}(HB_0)/\Gamma_0$ of M is called the maximal Margulis neighborhood of the cusp e.

Since the convergence or divergence of the Poincaré series does not depend on the base point x_0, we may assume that x_0 belongs to $H_0 \cap C\Lambda(\Gamma)$. Since the convergence or divergence of the relative Poincaré series does not depend on the choice of the horosphere, we will use this H_0 in the expression of $P_0(s)$ in all that follows.

Any rational geodesic r has a lift starting from ξ_0, which is unique modulo the action of Γ_0. The endpoint of any such lift is the center of an horosphere γH_0 for some γ in Γ. It follows from its definition that the depth of r is $d(H_0, \gamma H_0)$. We proved in [6, Lemma 2.7] that the map $r \mapsto \Gamma_0\gamma\Gamma_0$ from the set of rational geodesics to the set of double cosets $\Gamma_0\backslash\Gamma/\Gamma_0$ is a bijection. In particular the number $\mathcal{N}_e(x)$ of rational geodesics with depth at most x is

$$\mathcal{N}_e(x) = \mathrm{Card}\{\Gamma_0\gamma\Gamma_0 \in \Gamma_0\backslash\Gamma/\Gamma_0 \mid d(H_0, \gamma H_0) \leq x\}.$$

The fact that the relative Poincaré series $P_0(s)$ converges for $s > \limsup_{x\to+\infty} \frac{\log \mathcal{N}_e(x)}{x}$ and diverges if $s < \limsup_{x\to+\infty} \frac{\log \mathcal{N}_e(x)}{x}$ is then easily seen. In particular, Theorem 1.1 follows from Theorem 1.2.

Let $\mathcal{O}$ be the ring of integers of a number field K, where K is either $\mathbb{Q}$ or an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, with d a positive square free integer. We use the upper half-space models for the real hyperbolic spaces. Consider the cusp e in the orbifolds $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$ if $K = \mathbb{Q}$, or $\mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O})$ otherwise, corresponding to the parabolic fixed point $+\infty$. We proved in [6, Section 2.3] (with the obvious adaptation to the case of orbifolds) that the rational lines r are in one-to-one correspondence with the fractions $\frac{p}{q}$ modulo the additive group $\mathcal{O}$, with the depth of r being $\log |q|^2$, if this fraction is written with relatively prime numerator and denominator. Hence Corollary 1.3 follows from Theorem 1.1.

3 Proofs

With the notations and assumptions of the previous section, we start with two lemmas.

Lemma 3.1 *There exists a constant $C_1 \geq 0$, such that every double coset $\Gamma_0\gamma\Gamma_0$ in $\Gamma_0\backslash\Gamma/\Gamma_0$ has a representative γ which satisfies*

$$|d(H_0, \gamma H_0) - d(x_0, \gamma x_0)| \leq C_1.$$

Proof. Choose the identity as the representative of the trivial double coset. Let γ be in $\Gamma - \Gamma_0$. Let p_0 in H_0 and p_1 in γH_0 be such that the segment $[p_0, p_1]$ is the (unique) common perpendicular to H_0 and γH_0. In particular, $d(H_0, \gamma H_0) = d(p_0, p_1)$, and p_0, p_1 lie on the geodesic line between the centers of H_0 and γH_0, so that p_0, p_1 both belong to the Γ_0-invariant subset $H_0 \cap C\Lambda(\Gamma)$. Since ξ_0 is a bounded parabolic fixed point, the quotient $(H_0 \cap C\Lambda(\Gamma))/\Gamma_0$ is compact, hence has diameter bounded by $C_1' \geq 0$.

Since x_0 belongs to $H_0 \cap C\Lambda(\Gamma)$, there exists α in Γ_0 so that $d(p_0, \alpha x_0) \leq C_1'$, and β in Γ_0 so that $d(\gamma^{-1} p_1, \beta x_0) \leq C_1'$.

Since αx_0 lies on H_0 and $\gamma \beta x_0$ on γH_0, we have

$$d(H_0, \gamma H_0) = d(p_0, p_1) \leq d(\alpha x_0, \gamma \beta x_0).$$

Conversely, by the triangular inequality,

$$d(\alpha x_0, \gamma \beta x_0) \leq d(\alpha x_0, p_0) + d(p_0, p_1) + d(p_1, \gamma \beta x_0) \leq 2C_1' + d(H_0, \gamma H_0).$$

Hence the representative $\alpha^{-1}\gamma\beta$ of the double coset $\Gamma_0 \gamma \Gamma_0$ satisfies the condition of the Lemma with $C_1 = 2C_1'$. **3.1**

Note that by discreteness, the set of representatives as in the Lemma of a given double coset is finite. From now on, we will denote by the same letter a double coset and such a representative.

The following proposition is well-known (see for example the proof of [4, Lemme 4]).

Lemma 3.2 *There exists a constant $C_2 \geq 0$ (depending only on the upperbound on the curvature of M) such that the following holds. Let H, H' be horospheres in $\widetilde{M}$ bounding disjoint horoballs. Let $[p, p']$ be the common perpendicular segment, with p in H and p' in H'. For every x in H and x' in H',*

$$|d(x, x') - (d(x, p) + d(p, p') + d(x, p))| \leq C_2.$$

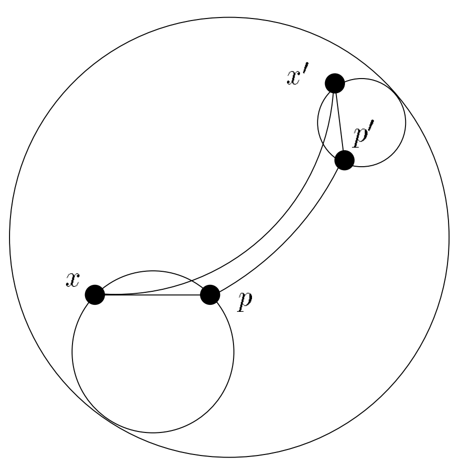

Figure 1 : The quasi-geodesic.

Proof. Up to replacing H, H' by inside concentric horospheres at distance 1, we may assume that $d(H, H') \geq 1$. By the convexity of horoballs, the piecewise geodesic $[x, p] \cup$

$[p, p'] \cup [p', x']$ has angles at least $\frac{\pi}{2}$ at p and at p'. Thus, since $d(p, p') \geq 1$, it is a quasi-geodesic, and the result follows for example from [5, Chapter 3]. **3.2**

Proof of Theorem 1.2 If f, g are maps from an interval I in $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$, write $f \asymp g$ if there exists a finite constant $c > 0$ such that $\frac{1}{c} g(s) \leq f(s) \leq c g(s)$ for all s in I. We write the Poincaré series as follows.

$$P(s) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0} \; \sum_{\alpha, \beta \in \Gamma_0} e^{-s\, d(x_0, \alpha\gamma\beta x_0)}.$$

Note that $d(x_0, \alpha\gamma\beta x_0) = d(\alpha^{-1} x_0, \gamma\beta x_0)$. The representatives γ of the (non trivial) double cosets have been chosen so that γx_0 lies at distance less than a constant C_1' from the endpoint on γH_0 of the common perpendicular segment between H_0 and γH_0. Applying Lemma 3.1 and Lemma 3.2, we obtain

$$P(s) \asymp \sum_{\gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0} \; \sum_{\alpha, \beta \in \Gamma_0} e^{-s\left(d(\alpha^{-1} x_0, x_0) + d(H_0, \gamma H_0) + d(\gamma x_0, \gamma\beta x_0)\right)}$$

$$= \left(\sum_{\gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0} e^{-s\, d(H_0, \gamma H_0)} \right) \left(\sum_{\alpha \in \Gamma_0} e^{-s\, d(x_0, \alpha x_0)} \right)^2 .$$

If $s > \delta_{\Gamma_0}$, then the Poincaré series of Γ_0 converges at s. Hence $P \asymp P_0$ on the interval $]\delta_{\Gamma_0}, +\infty[$. Theorem 1.2 follows. **1.2**

4 The case of Bianchi groups

For all undefined objects and unproved results in what follows, see [7] for instance.

Let K be a number field, $N = N_K$ the absolute norm on K, ζ_K the Dedekind zeta function and $\mathrm{Res}_K = \mathrm{Res}(\zeta_K, s = 1)$ the residue of ζ_K at the unity, $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K, $\mathcal{O}^*$ the units of $\mathcal{O}$, h_K the class number of K, W_K the number of roots of unity in K, R_K the regulator of K, and D_K the discriminant of K. Let μ be the Möbius function on the multiplicative semigroup of non zero ideals of $\mathcal{O}$, defined by $\mu(\mathcal{O}) = 1$, $\mu(I) = (-1)^k$ if the ideal I in $\mathcal{O}$ is the product of k distinct prime ideals, and $\mu(I) = 0$ if I is divisible by the square of a prime ideal. It satisfies $\sum_{I \mid J} \mu(I) = 1$ if $J = \mathcal{O}$, and 0 otherwise. Assume that $\mathcal{O}^*$ is finite (K is $\mathbb{Q}$ or an imaginary quadratic field), so that $\mathrm{Card}(\mathcal{O}^*) = W_K$ and $R_K = 1$.

For n a positive integer, define

$$\phi(x) = \phi_{\mathcal{O}}(x) = \ \mathrm{Card}\left\{ \frac{p}{q} \bmod \mathcal{O}, (p, q) \in \mathcal{O} \times (\mathcal{O} - \{0\}), (p, q) = 1 \text{ and } N(q) \leq x \right\}.$$

The following proposition might be already known. For the sake of completeness, we include its proof here.

Proposition 4.1 *If* $\epsilon = 1/[K : \mathbb{Q}] < 1$, *then*

$$\phi(x) = \frac{\mathrm{Res}_K}{2\, h_K\, \zeta_K(2)}\, x^2 + O(x^{2-\epsilon}).$$

Proof Since two irreducible fractions are equal if and only if the numerators and denominators are multiplied by the same unit, one has

$$\phi(x) = \frac{1}{W_K} \text{ Card}\{(q, p \bmod (q)), q \neq 0, (p,q) = 1 \text{ and } N(q) \leq x\}.$$

Hence $W_K \phi(x)$ is equal to

$$\sum_{(q,p \bmod (q)), q\neq 0, N(q)\leq x, (p,q)=1} 1 = \sum_{(q,p \bmod (q)), q\neq 0, N(q)\leq x} \sum_{I\,|\,(p,q)} \mu(I) = \sum_I \mu(I) f(I)$$

where I ranges over the ideals of $\mathcal{O}$ and

$$f(I) = \sum_{(q,p \bmod (q)), q\neq 0, I\,|\,(p,q), N(q)\leq x} 1 = \sum_{q\in I, q\neq 0, N(q)\leq x} \sum_{p\in I/(q)} 1 = W_K \sum_{(q)\subset I, N(q)\leq x} \frac{N(q)}{N(I)}$$

since $N(J) = \text{Card } \mathcal{O}/J$, where (q) is a non zero principal ideal, and since a generator of a principal ideal is uniquely defined up to units.

Lemma 4.2 *If* $S(x) = \text{Card}\{(q) \subset I, N(q) \leq x\}$*, then* $S(x) = \frac{\text{Res}_K}{h_K N(I)} x + O((\frac{x}{N(I)})^{1-\epsilon})$.

Proof. Note that $(q) \subset I$ if and only if $(q) = IJ$ for some ideal J in $\mathcal{O}$, and $N(IJ) = N(I)N(J)$. Hence

$$S(x) = \sum_{J\in[I]^{-1}, N(J)\leq \frac{x}{N(I)}} 1$$

where $[I]^{-1}$ is the inverse of the class of I in the class group. The result then follows from [7, theo. 7.6 page 361] (for example) for the main term and from [8] for the error term. **4.2**

Lemma 4.3 *If* $T(x) = \sum_{(q)\subset I, N(q)\leq x} N(q)$*, then* $T(x) = \frac{\text{Res}_K}{2h_K N(I)} x^2 + O(\frac{x^{2-\epsilon}}{N(I)^{1-\epsilon}})$.

Proof. This is immediate by applying Fubini's theorem and using the previous lemma. **4.3**

Now, since $\zeta_K(2-\epsilon)$ converges for $\epsilon < 1$, and since by [7, page 326],

$$\sum_I \frac{\mu(I)}{N^s(I)} = \frac{1}{\zeta_K(s)}$$

whenever $\Re(s) > 1$, one gets from Lemma 4.3

$$\phi(x) = \sum_I \mu(I) \frac{\text{Res}_K\ x^2}{2h_K N^2(I)} + O(x^{2-\epsilon}) = \frac{\text{Res}_K}{2\, h_K\, \zeta_K(2)}\ x^2 + O(x^{2-\epsilon})$$

if $\epsilon < 1$. **4.1**

Since $\text{Res}_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{W_k \sqrt{|D_K|}}$, one has for $K = \mathbb{Q}(\sqrt{-d})$, with $D_K = d$ if $d \equiv 3 \bmod 4$ and $D_K = 4d$ otherwise, with $w = 4, 6$ if $d = 1, 3$ and $w = 2$ otherwise,

$$\phi_{\mathcal{O}_{-d}}(x) = \frac{\pi}{w\, \zeta_{\mathbb{Q}(\sqrt{-d})}(2) \sqrt{D_K}}\ x^2 + O(x^{\frac{3}{2}}).$$

For $K = \mathbb{Q}$, an analogous but much simpler proof gives the well-known result

$$\phi_{\mathbb{Z}}(x) = \frac{3}{\pi^2}x^2 + O(x \log x)$$

(see [9, page 45] for instance, and [10] for an improvement on the log exponent).

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