UBIQUITY OF GEOMETRIC FINITENESS IN BOUNDARIES OF DEFORMATION SPACES OF HYPERBOLIC 3-MANIFOLDS

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Abstract. We show that geometrically finite Kleinian groups are dense in the boundary of the quasiconformal deformation space of any geometrically finite Kleinian group.

1. Introduction. In this paper we prove that geometrically finite hyperbolic manifolds are dense in the boundary of the quasiconformal deformation space of any geometrically finite hyperbolic 3-manifold. Our main result generalizes the fact, established in [17], that maximal cusps are dense in the boundaries of quasiconformal deformation spaces of hyperbolic 3-manifolds with connected conformal boundary. Both this paper and its predecessor [17] make central use of techniques developed by McMullen in his proof that maximal cusps are dense in the boundary of any Bers Slice [35].

It will be convenient to formalize the statement of our main result using the language of pared manifolds. A *pared manifold* (M, P) consists of a compact, irreducible, oriented 3-manifold M and a collection P of disjoint incompressible annuli and tori in ∂M such that:

(1) If A is an abelian subgroup of $\pi_1(M)$ which is not cyclic, then A is conjugate into the fundamental group of a component of P, and

(2) every map $\phi : (S^1 \times I, S^1 \times \partial I) \to (M, P)$ that is injective on the fundamental groups, is homotopic, as a map of pairs, into P.

If G is a torsion-free group and $\rho : G \to \mathbf{PSL}_2(\mathbf{C})$ is a discrete faithful representation, then $N_\rho = \mathbf{H}^3/\rho(G)$ is a hyperbolic 3-manifold. The domain of discontinuity $\Omega(\rho)$ of $\rho(G)$ is the largest open subset of $\widehat{\mathbf{C}}$ on which $\rho(G)$ acts properly discontinuously. The quotient $\partial_c N_\rho = \Omega(\rho)/\rho(G)$ is the *conformal boundary* of N_ρ. We will say that ρ is *geometrically finite* if $N_\rho \cup \partial_c N_\rho$ is homeomorphic to $M_\rho - P_\rho$, where (M_ρ, P_ρ) is a pared 3-manifold. (For a discussion of several equivalent definitions of geometric finiteness, see Bowditch [10].) We say that ρ is a *maximal cusp* if ρ is geometrically finite and every component of $M_\rho - P_\rho$

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is a thrice punctured sphere. (In McMullen [35] a maximal cusp in the boundary of a Bers Slice has a different, but analogous, definition.)

A discrete faithful representation $\rho : \pi_1(M) \to \mathbf{PSL}_2(\mathbf{C})$ is a *geometrically finite uniformization* of the pared 3-manifold (M, P) if there exists an orientation-preserving homeomorphism $h : M - P \to N_\rho \cup \partial_c N_\rho$ in the homotopy class determined by ρ. Let $GF_0(M, P)$ denote the space of (conjugacy classes of) geometrically finite uniformizations of (M, P). If P is empty, then $GF_0(M, \emptyset)$ is sometimes denoted $CC_0(M)$, as in [17]. If $\rho_0 \in GF_0(M, P)$, then $GF_0(M, P)$ is the space of all (conjugacy classes of) representations quasiconformally conjugate to ρ_0. Thurston's geometrization theorem (see Morgan [37] for a statement in the language of pared manifolds) implies that $GF_0(M, P)$ is always nonempty if ∂M is nonempty.

The space $GF_0(M, P)$ naturally sits as a subset of the space $AH(\pi_1(M), \pi_1(P))$ of all (conjugacy classes of) discrete faithful representations $\rho : \pi_1(M) \to \mathbf{PSL}_2(\mathbf{C})$ such that $\rho(g)$ is parabolic if g is conjugate to an element of $\pi_1(P)$. If P consists only of the toroidal boundary components of M, then $AH(\pi_1(M), \pi_1(P))$ is simply denoted by $AH(\pi_1(M))$. One may think of $AH(\pi_1(M), \pi_1(P))$ as the space of all (marked) hyperbolic 3-manifolds homotopy equivalent to M which have cusps in the homotopy classes associated to components of P.

In this language our main theorem becomes:

MAIN THEOREM. *Let (M, P) be a pared 3-manifold such that $\pi_1(M)$ is nonabelian and $\partial M - P$ is nonempty. Then conjugacy classes of geometrically finite representations are dense in the boundary of $GF_0(M, P)$.*

Remark. If $\partial M = P$, then Mostow-Prasad rigidity [39, 40] implies that $GF_0(M, P) = AH(\pi_1(M), \pi_1(P))$ consists of at most two points. If $\pi_1(M)$ is abelian, all the representations in $AH(\pi_1(M), \pi_1(P))$ are elementary and the space can be described quite explicitly.

One nearly immediate corollary of our result is that manifolds with arbitrarily short geodesics are topologically generic in the boundary of $GF_0(M, P)$.

COROLLARY A. *Let (M, P) be a pared 3-manifold with nonabelian fundamental group such that $\partial M - P$ is nonempty. Then the set of conjugacy classes $[\rho] \in \partial GF_0(M, P)$ such that N_ρ contains arbitrarily short geodesics is a dense G_δ subset of $\partial GF_0(M, P)$.*

Motivation. Thurston's Ending Lamination conjecture provides a conjectural classification of the representations in $AH(\pi_1(M), \pi_1(P))$ and the geometrically finite representations correspond to the "rational points" in this classification, so we may think of our result as saying that rational points are dense in the boundary of $GF_0(M, P)$.

We illustrate this analogy with the example of the space of punctured torus groups. In this case $(M, P) = (F \times I, \partial F \times I)$ where F is a punctured torus.

Bers' Simultaneous Uniformization Theorem [6] implies that $GF_0(M,P)$ may be parameterized by $\mathcal{T}(F) \times \mathcal{T}(F)$. The Teichmüller space $\mathcal{T}(F)$ of all finite area hyperbolic structures on F may be identified with $\mathbf{H}^2$ and the "boundary" of $\mathbf{H}^2$ is identified with $\widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. Minsky [36] proved that one may extend Bers' identification of $GF_0(M,P)$ with $\mathbf{H}^2 \times \mathbf{H}^2$ to a one-to-one correspondence between $AH(\pi_1(M), \pi_1(P))$ and $\overline{\mathbf{H}^2} \times \overline{\mathbf{H}^2} - \Delta$ where Δ denotes the diagonal in $\partial\mathbf{H}^2 \times \partial\mathbf{H}^2$. In this correspondence, a representation corresponding to a point in $\overline{\mathbf{H}^2} \times \overline{\mathbf{H}^2} - \Delta$ is geometrically finite if and only if both of its coordinates lie in $\mathbf{H}^2 \cup \widehat{\mathbf{Q}}$. Although this correspondence is geometrically natural, it is also known not to be a homeomorphism (see section 12.3 of Minsky [36]).

Let $GF(\pi_1(M), \pi_1(P))$ be the space of minimally parabolic, geometrically finite representations in $AH(\pi_1(M), \pi_1(P))$. (A representation $[\rho] \in AH(\pi_1(M), \pi_1(P))$ is *minimally parabolic* if $\rho(g)$ is parabolic only if g is conjugate to an element of $\pi_1(P)$.) Marden [29] and Sullivan [43] proved that $GF(\pi_1(M), \pi_1(P))$ is the interior of $AH(\pi_1(M), \pi_1(P))$ as a subset of the character variety $X(\pi_1(M), \pi_1(P))$. Yair Minsky (with coauthors Brock and Canary) has recently announced a proof of Thurston's Ending Lamination Conjecture for manifolds in $AH(\pi_1(M), \pi_1(P))$ in the case that $\partial M - P$ is incompressible. This result implies that $GF(\pi_1(M), \pi_1(P))$ is dense in $AH(\pi_1(M), \pi_1(P))$ when $\partial M - P$ is incompressible. (Bromberg [15] and Brock-Bromberg [13] had previously established that minimally parabolic representations in $AH(\pi_1(M), \pi_1(P))$ lie in the boundary of $GF(\pi_1(M), \pi_1(P))$ if ∂M is incompressible and P consists only of tori.) Since every point in $GF(\pi_1(M), \pi_1(P))$ lies in $GF_0(M', P')$ for some pared manifold (M', P'), we may combine our result with Brock, Canary and Minsky's result to obtain:

COROLLARY B. *Let (M,P) be a pared 3-manifold such that every component of $\partial M - P$ is incompressible. Then conjugacy classes of geometrically finite representations are dense in the boundary of $AH(\pi_1(M), \pi_1(P))$.*

Remark. In a remark at the end of Section 3 we explain how one may use Brock, Canary and Minsky's result in place of our key estimate (Theorem 9.1) to give a different proof of our Main Theorem in the case that $\partial M - P$ has incompressible boundary.

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2. Background. In this section we collect background results which will be useful in the proof. We are then able to give an outline of the proof in Section 3.

2.1. Representation spaces and their metrics. Let (M, P) be a pared 3-manifold such that $\pi_1(M)$ is nonabelian. We let $\mathcal{R}(\pi_1(M), \pi_1(P))$ denote the set of representations $\rho : \pi_1(M) \to \mathbf{PSL}_2(\mathbf{C})$ such that $\rho(g)$ is either parabolic or the identity element if g is conjugate to an element of $\pi_1(P)$. Let $\mathcal{D}(\pi_1(M), \pi_1(P))$ be the closed subset of discrete faithful representations in $\mathcal{R}(\pi_1(M), \pi_1(P))$. We give both $\mathcal{D}(\pi_1(M), \pi_1(P))$ and $\mathcal{R}(\pi_1(M), \pi_1(P))$ the compact-open topology.

The space $AH(\pi_1(M), \pi_1(P))$ is simply the quotient of $\mathcal{D}(\pi_1(M), \pi_1(P))$ by $\mathbf{PSL}_2(\mathbf{C})$, acting by conjugation. It sits naturally as a subset of the character variety $X(\pi_1(M)), \pi_1(P))$ which is the algebreo-geometric quotient of $\mathcal{R}(\pi_1(M), \pi_1(P))$ by $\mathbf{PSL}_2(\mathbf{C})$, see Morgan-Shalen [38] for details.

One may find a finite collection of elements of $\pi_1(M)$ whose squared traces give rise to a proper embedding of $AH(\pi_1(M), \pi_1(P))$ into $\mathbf{C}^m$ for some m (see, for example Proposition 2.2 in [17]). If $g \in \pi_1(M)$, let $\tau_g(\rho)$ denote the square of the trace of $\rho(g)$, then τ_g is a well defined continuous function on $\mathcal{R}(\pi_1(M), \pi_1(P))$. Since τ_g is invariant under conjugation, it descends to a continuous function $\bar{\tau}_g : AH(\pi_1(M), \pi_1(P)) \to \mathbf{C}$.

PROPOSITION 2.1. *Let M be a compact, orientable, irreducible, atoroidal 3-manifold whose boundary has a nontorus component. Then there exists a finite set $\{a_1, \dots, a_m\}$ of primitive elements of $\pi_1(M)$ such that:*

(1) *if $[\rho_1], [\rho_2] \in AH(\pi_1(M))$ and $\bar{\tau}_{a_i}([\rho_1]) = \bar{\tau}_{a_i}([\rho_2])$ for all $i = 1, \dots, m$, then $[\rho_1] = [\rho_2]$; and*

(2) *given any $K > 0$, the set*

$$\{[\rho] \in AH(\pi_1(M)) \mid \sum_{i=1}^{m} |\bar{\tau}_{a_i}(\rho)| \leq K\}$$

is compact.

If $\mathcal{A} = \{a_1, \dots, a_m\}$ is a collection of primitive elements of $\pi_1(M)$ which satisfies conditions (1) and (2) of Proposition 2.1 then we call $\mathcal{A}$ an *allowable collection of test elements*. The map $\bar{\tau} : AH(\pi_1(M), \pi_1(P)) \to \mathbf{C}^m$ given by $\bar{\tau}(\rho) = (\bar{\tau}_{a_1}(\rho), \dots, \bar{\tau}_{a_m}(\rho))$ is a proper embedding of $AH(\pi_1(M), \pi_1(P))$ into $\mathbf{C}^m$. Let $d_{\mathcal{A}}$ be the metric on $AH(\pi_1(M), \pi_1(P))$ which it inherits as a subset of $\mathbf{C}^m$. Explicitly,

$$d_{\mathcal{A}}([\rho_1], [\rho_2]) = \sqrt{\sum_{i=1}^{m} |\bar{\tau}_{a_i}([\rho_1]) - \bar{\tau}_{a_i}([\rho_2])|^2}.$$

2.2. Cores of 3-manifolds. If $N = \mathbf{H}^n/\Gamma$ is a hyperbolic manifold, then the *ϵ-thin part* $N_{thin(\epsilon)}$ is the set of points in N of injectivity radius less than ϵ. There exists $\mu > 0$ (called the Margulis constant) such that if N is any hyperbolic 3-manifold and $\epsilon < \mu$, then any unbounded component of $N_{thin(\epsilon)}$ is the quotient of a horoball in $\mathbf{H}^3$ by a parabolic subgroup of Γ. In particular, each unbounded component of $N_{thin(\epsilon)}$ is homeomorphic to $S^1 \times \mathbf{R} \times (0, \infty)$ or $S^1 \times S^1 \times (0, \infty)$. The components homeomorphic to $S^1 \times \mathbf{R} \times (0, \infty)$ are called *rank one cusps* and the components homeomorphic to $S^1 \times S^1 \times (0, \infty)$ are called rank two cusps. (See Chapter D of Benedetti-Petronio [5] for details.)

It will be useful in the proof of Proposition 6.1 to bound the growth of injectivity radius as one travels away from a point in a cusp. Suppose that x lies in a cusp of $N_{thin(\mu)}$. Then there exists a lift $\widetilde{x}$ of x to $\mathbf{H}^3$ and a parabolic element γ of Γ such that $d(\widetilde{x}, \gamma(\widetilde{x})) = 2\,\mathrm{inj}_N(x)$. One may then check (see, for example, part (iii) of Theorem 7.35.1 of Beardon [4]) that if $d(\widetilde{y}, \widetilde{x}) = K$, then

$$d(\widetilde{y}, \gamma(\widetilde{y})) < \sinh d(\widetilde{y}, \gamma(\widetilde{y})) \leq e^K \sinh\left(d(\widetilde{x}, \gamma(\widetilde{x}))\right).$$

Therefore, if $y \in N$, x lies in a cusp of $N_{thin(\mu)}$ and $d(x, y) \leq K$, then

$$\mathrm{inj}_N(y) \leq \frac{e^K}{2} \sinh\left(2\ \mathrm{inj}_N(x)\right) \tag{1}$$

If $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold, its convex core $C(N)$ is the quotient of the convex hull $CH(\Lambda(\Gamma))$ of its limit set $\Lambda(\Gamma)$ by Γ. The nearest point retraction $r : N \to C(N)$ takes each point in N to the nearest point in $C(N)$. It extends continuously to a map $\bar{r} : N \cup \partial_c N \to C(N)$. If Γ is finitely generated and $\Lambda(\Gamma)$ does not lie in a circle in $\widehat{\mathbf{C}}$, then $\bar{r}$ is properly homotopic to a homeomorphism. (See Epstein-Marden [19] or Thurston [44] for an extensive discussion of the nearest point retraction and the convex core.)

Canary [16] showed that an upper bound on the length of a closed curve in $\partial_c N$ gives an upper bound on the length of the corresponding closed geodesic in $\partial C(N)$. (The conformal boundary $\partial_c N$ is always given its associated Poincaré metric in this paper.) We recall, see Epstein-Marden [19], that $\partial C(N)$ is a complete hyperbolic surface in its intrinsic metric.

THEOREM 2.2. *Let N be a hyperbolic 3-manifold with finitely generated fundamental group and let γ be a closed geodesic of length L in $\partial_c N$, then*

$$l_{\partial C(N)}(\bar{r}(\gamma)^*) \leq 45 L e^{\frac{L}{2}}$$

where $\bar{r} : N \cup \partial_c N \to C(N)$ is the nearest point retraction and $\bar{r}(\gamma)^*$ is the closed geodesic in $\partial C(N)$ in the homotopy class of $r(\gamma)$.

If $\epsilon < \mu$, then we let N^0_ϵ be obtained from N by removing the unbounded components of its ϵ-thin part. Let $C(N)_\epsilon = C(N) \cap N^0_\epsilon$ and $P(N)_\epsilon = C(N) \cap \partial N^0_\epsilon$. If N is geometrically finite and $\Lambda(\Gamma)$ does not lie in the circle, then $C(N) - C(N)_\epsilon$ is homeomorphic to $P(N)_\epsilon \times (0, \infty)$ (for any $\epsilon < \mu$) and $(C(N)_\epsilon, P(N)_\epsilon)$ is a pared manifold, see section 6 of Morgan [37].

A compact core C for a manifold D will be a compact submanifold such that the inclusion map is a homotopy equivalence. (When one is not working in the setting of aspherical manifolds one sometimes only requires that the inclusion map induce an isomorphism on the fundamental groups.) We say that (M,P) is a relative compact core for N_ϵ^0 (or for $C(N)_\epsilon$) if M is a compact core for N_ϵ^0 (or for $C(N)_\epsilon$) and M intersects each component of ∂N_ϵ^0 in a connected submanifold which is a compact core for that component. It follows from work of McCullough [34] and Kulkarni-Shalen [28] that N_ϵ^0 and $C(N)_\epsilon$ have relative compact cores if $\pi_1(N)$ is finitely generated. Moreover, if R is a compact submanifold of $\partial C(N) \cap C(N)_\epsilon$, then there exists a relative compact core for $C(N)_\epsilon$ such that $M \cap (\partial C(N) \cap C(N)_\epsilon) = R$ (see McCullough [34].)

Proposition 1.3 of Bonahon [9] asserts that the ends of N_ϵ^0 are in one-to-one correspondence with the components of $\partial M - P$ if (M,P) is a relative compact core for N_ϵ^0. (Although Proposition 1.3 is stated only for the case where $\partial M - P$ is incompressible, the proof goes through in the same manner in the general case.) In particular, each component of $N_\epsilon^0 - M$ is a neighborhood of exactly one end of N_ϵ^0 and each end of N_ϵ^0 has a neighborhood of this form. An end is said to be *geometrically finite* if the associated component of $N_\epsilon^0 - M$ contains a component of $N_\epsilon^0 - C(N_\rho)$. Otherwise, the end is said to be geometrically infinite. A hyperbolic 3-manifold N with finitely generated fundamental group is geometrically finite if and only if all the ends of $\widehat{N}_\epsilon^0$ are geometrically finite (see page 81 of Bonahon [9]).

It will be useful to note that ends associated to thrice-punctured spheres in $\partial M - P$ are geometrically finite.

LEMMA 2.3. *Suppose that N is a hyperbolic 3-manifold with finitely generated fundamental group, $\epsilon < \mu$, and (M,P) is a relative compact core for N_ϵ^0. If a component F of $\partial M - P$ is a thrice punctured sphere, then the associated end of N_ϵ^0 is geometrically finite.*

We first recall (see Lemma 2.4 below) that a free Kleinian group generated by two parabolic elements whose product is also parabolic is necessarily Fuchsian, i.e., preserves a round disk in $\widehat{\mathbf{C}}$. See, for example, Theorem IX.C.1 of Maskit [32]. (Note that although Maskit assumes in his statement that these are the only conjugacy classes of parabolics, this is not used in the proof.)

LEMMA 2.4. *Let Γ be a Kleinian group generated by parabolic elements γ_1 and γ_2 such that $\gamma_1\gamma_2$ is parabolic. Then Γ is a Fuchsian group.*

Proof of Lemma 2.3. The surface F must be incompressible since each simple closed curve in Z is homotopic to a core curve of a component of P, and hence homotopically nontrivial in M. If $N = \mathbf{H}^3/\Gamma$, then the subgroup of Γ associated to $\pi_1(F)$ is freely generated by two parabolic elements whose product is parabolic, so the subgroup must be Fuchsian, and the end associated to F must be geometrically finite. $\square$

2.3. Pared manifolds and pinchable collections of curves. We recall that (X, Y) is a 3-manifold pair if X is a 3-manifold and Y is a subsurface of ∂X. The pair (X, Y) is said to be compact and irreducible if X is compact and irreducible. We will say that the pair (X, Y) has *relatively incompressible boundary* if every component of $\partial X - Y$ is incompressible in X.

We say that a pared manifold (X, Y) is *acylindrical* if it has relatively incompressible boundary and every incompressible embedded torus in X or properly embedded incompressible annulus in X with boundary in $\partial X - Y$ is parallel into $\partial X - Y$ or parallel into Y. (A properly embedded surface W such that $\partial W \subset \partial X - Y$ is *parallel into* $\partial X - Y$, if there exists a subsurface W' of $\partial X - Y$ such that W and W' bound a region in X homeomorphic to $W \times I$ such that W is identified with $W \times \{0\}$, W' is identified with $W \times \{1\}$ and $\partial W \times [0, 1) \subset \partial X - Y$. Similarly, W is *parallel into* Y if there exists a component W' of Y which bounds a region in X of the same form as above.)

It is a consequence of Johannson's Classification theorem [25]—see Lemma X.23 in Jaco [24] for a statement in language similar to ours—that any homotopy equivalence of pairs between an acylindrical pared manifold and a compact, irreducible manifold pair with relatively incompressible boundary is homotopic to a homeomorphism of pairs.

THEOREM 2.5. *Let (M, P) be an acylindrical pared 3-manifold and $(\widehat{M}, \widehat{P})$ a manifold pair with relatively incompressible boundary. Suppose that $f : (M, P) \to (\widehat{M}, \widehat{P})$ is a map of pairs, f is a homotopy equivalence, and $f|_P : P \to \widehat{P}$ is a homeomorphism, then f is homotopic to a pared homeomorphism.*

We say that a pared manifold (M, P) is *maximal* if each component of $\partial M - P$ is a thrice punctured sphere. The observation that a maximal pared manifold is acylindrical plays a key role in the proof of Proposition 8.1.

LEMMA 2.6. *If (M, P) is a maximal pared manifold, then (M, P) is acylindrical.*

Proof of Lemma 2.6. If (M, P) did not have relatively incompressible boundary then there would be a homotopically nontrivial simple closed curve α in $\partial M - P$ (by Dehn's Lemma) which is homotopically trivial in M. But, since each component of $\partial M - P$ is a thrice punctured sphere, α must be homotopic to the core curve of an annular component of P and hence cannot be homotopically trivial in M.

If W is an incompressible torus in M, then $\pi_1(W)$ is identified with a rank two free abelian subgroup of $\pi_1(M)$, so W is homotopic into a toroidal component P_0 of P. Since W and P_0 are both embedded, they must bound a product region, so W is parallel to P_0 (see Lemma 5.3 in Waldhausen [47]).

If W is a properly embedded incompressible annulus with boundary in $\partial M - P$, then each component of ∂W is parallel to the core curve of a component of P (since each component of $\partial M - P$ is a thrice punctured sphere). Therefore, W is properly homotopic to an annular component P_0 of P. It follows that the boundary components of W bound an annulus A in ∂M and that A either contains P_0 or is disjoint from it. If A contains P_0, then W is parallel to P_0. If A is contained in $\partial M - P$, then W is parallel to a subannulus of A, which lies in $\partial M - P$. (Again we are applying Lemma 5.3 in [47].) □

Suppose that $[\rho] \in GF_0(M,P)$ and $h : M - P \to N_\rho \cup \partial_c N_\rho$ is an orientation-preserving homeomorphism. If C is a collection of disjoint simple closed geodesics in $\partial_c N_\rho$, then let $\mathcal{N}(h^{-1}(C))$ be a closed regular neighborhood of $h^{-1}(C)$ in $\partial M - P$. We say that C is *pinchable* if $(M, P \cup \mathcal{N}(h^{-1}(C)))$ is a pared 3-manifold.

If $\rho \in GF_0(M,P)$ and $C = \{c_1, \ldots, c_m\}$ is a pinchable collection of curves in $\partial_c N_\rho$ then each element of C is associated to the conjugacy class of a nontrivial element of $\rho(\pi_1(M))$ since each component of $\mathcal{N}(h^{-1}(C))$ is incompressible. No two elements of C are associated to the same conjugacy class, since then there would be an essential annulus joining distinct components of $\mathcal{N}(h^{-1}(C))$, which would violate the assumption that $(M, P \cup \mathcal{N}(h^{-1}(C)))$ is a pared 3-manifold. Similarly, no element of $\rho(\pi_1(M))$ which represents a curve in C, can be conjugate to an element of $\rho(\pi_1(P))$. Since elements of $\rho(\pi_1(M))$ are parabolic if and only if they are conjugate to elements of $\rho(\pi_1(P))$, this implies that each conjugacy class associated to a curve in C consists of hyperbolic elements. Each element in the conjugacy class determined by an element of C is primitive, since otherwise there would be an essential annulus in $(M, P \cup \mathcal{N}(h^{-1}(C)))$ with both boundary components lying in the same component of $\mathcal{N}(h^{-1}(C))$ (see, for example, Lemma 5.1.1 in Canary-McCullough [18]).

2.4. Analysis on $GF_0(M,P)$. Let X be a finite type Riemann surface. The Teichmüller space $\mathcal{T}(X)$ of all marked Riemann surfaces which are quasiconformally homeomorphic to X can be identified with the quotient $B_1(X)/\mathcal{Q}_0(X)$, where $B_1(X)$ denotes the space of Beltrami differentials of L^∞-norm less than 1 and $\mathcal{Q}_0(X)$ denotes the group of all quasiconformal self-homeomorphisms of X that are homotopic to the identity. Let $\Phi_X : B_1(X) \to \mathcal{T}(X)$ be the quotient map. The tangent space to Teichmüller space at the basepoint can then be identified with $B(X)/N(X)$ where $B(X)$ is the space of all Beltrami differentials and $N(X)$ is the space of infinitesmally trivial Beltrami differentials. Let $D\Phi_X : B(X) \to T_X\mathcal{T}(X)$ be the projection map. (See Gardiner [22] or Imayoshi-Taniguchi [23] for more details on the analytic theory of Teichmüller space.)

The Teichmüller space $\mathcal{T}(X)$ can also be thought of as the space of pairs (Y,f) where Y is a Riemann surface and $f : X \to Y$ is a quasiconformal homeomorphism, and where two pairs (Y_1,f_1) and (Y_2,f_2) are equivalent if there exists a conformal map $g : Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$. The pair (Y,f) can be identified with the equivalence class of the Beltrami differential $(f_{\bar z}/f_z)\frac{d\bar z}{dz}$ of f.

If $\beta : [0, B] \to \mathcal{T}(X)$ is a differentiable path and $\beta(t) = (X_t, g_t)$, then for each $t \in [0, B]$ there is a projection map $\Phi_t : B_1(X_t) \to \mathcal{T}(X) \cong B_1(X_t)/\mathcal{Q}_0(X_t)$ such that $\Phi_t(0) = \beta(t)$. The tangent vector $\beta'(t)$ then lies in $D\Phi_t(B(X_t))$ which may be identified with the tangent space to $\mathcal{T}(X)$ at $\beta(t)$.

We note that it is well known that given any marked Riemann surface X and a short simple closed curve C on X, one may construct a (uniformly) bounded length path in Teichmüller space joining X to a Riemann surface Y where the length of C has been halved. It will be important for us that one may choose the path so that the tangent vectors are represented by Beltrami differentials supported on the thin part.

LEMMA 2.7. (Lemma 2.1 in [17]) *Let $L_0 > 0$ be given, let X be a finite area hyperbolic surface, and let γ be a simple closed geodesic on X of length $L \leq L_0$. There exists a positive number B depending only on L_0 and a path $\beta : [0, B] \to \mathcal{T}(X)$, with $\beta(t) = (X_t, g_t)$, such that the following conditions hold:*

(1) $\beta(0) = (X, id)$,

(2) *for all $t \in [0, B]$ we have $l_{X_t}(g_t(\gamma)) \leq L$,*

(3) $l_{X_B}(g_B(\gamma)) \leq \frac{L}{2}$, *and*

(4) *$\beta'(t)$ is represented, for all t, by $\mu_t \in B(X_t)$ such that $\|\mu_t\|_\infty \leq 1$ and μ_t is supported on the $2L$-thin part of X_t associated to the curve $g_t(\gamma)$.*

Two representations ρ_1 and ρ_2 in $\mathcal{D}(\pi_1(M), \pi_1(P))$ are said to be *quasiconformally conjugate* if there exists a quasiconformal map $\widetilde{\phi} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $\rho_2(g) = \widetilde{\phi}\rho_1(g)\widetilde{\phi}^{-1}$ for all $g \in \pi_1(M)$. In this case, $\widetilde{\phi}$ descends to an orientation-preserving homeomorphism $\phi : \partial_c N_{\rho_1} \to \partial_c N_{\rho_2}$ which itself extends to a homeomorphism between $N_{\rho_1} \cup \partial_c N_{\rho_1}$ and $N_{\rho_2} \cup \partial_c N_{\rho_2}$ in the homotopy class determined by $\rho_2 \circ \rho_1^{-1}$. In particular, $[\rho_1] \in GF_0(M, P)$ if and only if $[\rho_2] \in GF_0(M, P)$.

Let $[\rho_0] \in GF_0(M, P)$. If $[\rho] \in GF_0(M, P)$, then Marden's Isomorphism Theorem (Theorem 8.1 in [29]) implies that there exists a quasiconformal map $\widetilde{\phi}$ conjugating ρ_0 to ρ. The pair $(\partial_c N_\rho, \phi)$ may be thought of as a point in the Teichmüller space $\mathcal{T}(\partial_c N_{\rho_0})$. However, if we precompose ϕ by a quasiconformal self-map ψ of $\partial_c N_{\rho_0}$ which extends to a homeomorphism of $N_\rho \cup \partial_c N_{\rho_0}$ that is homotopic to the identity, then $(\partial_c N_\rho, \phi \circ \psi)$ is another point in $\mathcal{T}(\partial_c N_{\rho_0})$ which is naturally associated to ρ. Using work of Bers, Kra and Maskit (see Bers [7] or [18]) one may identify $GF_0(M, P)$ with $\mathcal{T}(\partial_c N_{\rho_0})/\mathrm{Mod}_0(\rho_0)$, where $\mathrm{Mod}_0(\rho_0)$ denotes the group of quasiconformal automorphisms of $\partial_c N_{\rho_0}$ which extend to maps of $N_{\rho_0} \cup \partial_c N_\rho$ that are homotopic to the identity. $\mathrm{Mod}_0(\rho_0)$ acts freely and properly discontinuously on $\mathcal{T}(\partial_c N_{\rho_0})$, so $GF_0(M, P)$ is a manifold (see Maskit [31]).

Since $\partial_c N_{\rho_0}$ is homeomorphic to $\partial M - P$ we may identify $\mathcal{T}(\partial_c N_{\rho_0})$ with $\mathcal{T}(\partial M - P)$ and $\mathrm{Mod}_0(\rho_0)$ with the group $\mathrm{Mod}_0(M, P)$ of isotopy classes of pared homeomorphisms of (M, P) that are homotopic to the identity. For the remainder of the paper, we will identify $GF_0(M, P)$ with $\mathcal{T}(\partial M - P)/\mathrm{Mod}_0(M, P)$ and let

$q_M : \mathcal{T}(\partial M - P) \to GF_0(M,P)$ denote the quotient map. With this identification, if $(Y,f) \in \mathcal{T}(\partial M - P)$ and $q_M(Y,f) = [\rho]$, then one may identify $\partial_c N_\rho$ with the Riemann surface Y and f extends to a homeomorphism $\bar{f} : M - P \to N_\rho \cup \partial_c N_\rho$ such that $[\bar{f}_*] = [\rho]$.

If a is any element of $\pi_1(M)$ which is not conjugate into $\pi_1(P)$, then there is a natural map $\Upsilon_a : GF_0(M,P) \to CC_0(S^1 \times D^2)$ given by $\Upsilon_a([\rho]) = [\rho_a]$ where ρ_a denotes the restriction of ρ to the cyclic subgroup $\langle a \rangle$ of $\pi_1(M)$ generated by a. We can identify $CC_0(S^1 \times D^2)$ with $\mathcal{T}(T^2)/\mathrm{Mod}_0(S^1 \times D^2)$. Notice that Υ_a lifts to a map $\widetilde{\Upsilon}_a : \mathcal{T}(\partial M - P) \to \mathcal{T}(T^2)$. (For our purposes, it will never matter which lift is chosen.)

Since $\mathrm{Mod}_0(M,P)$ and $\mathrm{Mod}_0(S^1 \times D^2)$ act freely, properly discontinuously and by isometries (of the Teichmüller metrics) on $\mathcal{T}(\partial M - P)$ and $\mathcal{T}(T^2)$, both $GF_0(M,P)$ and $CC_0(S^1 \times D^2)$ inherit a quotient Teichmüller metric. If $[\rho] \in GF_0(M,P)$, then we may naturally identify $\mathcal{T}(\partial M - P)$ with $B_1(\partial_c N_\rho)/\mathcal{Q}_0(\partial_c N_\rho)$. Let

$$\Phi_\rho : B_1(\partial_c N_\rho) \to \mathcal{T}(\partial M - P)$$

be a projection map so that $q_M(\Phi_\rho(0)) = [\rho]$ and let $\overline{\Phi}_\rho = q_M \circ \Phi_\rho$.

The main local estimate from the previous paper, Theorem 14.1 in [17], asserted that, on the infinitesimal level, a deformation supported on the thin part associated to a "short" curve on the conformal boundary $\partial_c N_\rho$ has a small effect on the complex length of a moderate length element of $\rho(\pi_1(M))$. One may combine this with Lemma 2.7 to see that one may halve the length of a short curve in the conformal boundary without appreciably changing the complex length of a moderate length element of $\rho(\pi_1(M))$. The proof carries over nearly word for word into our slightly more general setting. We denote by $l(\rho(a))$ the real translation length of $\rho(a)$.

THEOREM 2.8. *Given $d_0 > 0$, there exists $c_0 > 0$ and $K_0 > 0$ with the following properties. Suppose that (M,P) is a pared manifold, a is a primitive element in $\pi_1(M)$, $[\rho] \in GF_0(M,P)$ and $l(\rho(a)) > d_0$. Suppose that C is a pinchable collection of disjoint simple closed geodesics in $\partial_c N_\rho$, none of which represents $\rho(a)$, such that each element of C has length at most L where*

$$L \leq c_0 e^{-l(\rho(a))}.$$

If $\mu \in B(\partial_c N_\rho)$, $\|\mu\|_\infty \leq 1$, and μ is supported on the union of the $2L$-thin parts associated to elements of C, then

$$\|D\Upsilon_a(D\bar{\Phi}_\rho(\mu))\| \leq K_0 L.$$

3. Outline of proof. The proof of the main theorem follows the same rough outline as the proof of the main result in [17] (which was in turn similar to the outline of proof of the main result of McMullen [35]). However, new technical difficulties arise since we must be able to approximate points where the limit set is not the entire sphere.

Many of the difficulties associated with points where the limit set is not the sphere are resolved by applying a result of Evans [20]. If $\{\rho_n\}$ is a sequence in $\mathcal{D}(\pi_1(M), \pi_1(P))$ converging to a minimally parabolic representation ρ, then Evans proved that $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$ in the sense of Carathéodory. So, for all large enough n, we may divide $\Omega(\rho_n)$ into those components $\Omega_F(\rho_n)$ which converge to components of $\Omega(\rho)$ and those components $\Omega_D(\rho_n)$ which "degenerate." Let $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$ be the "degenerating portion" of the conformal boundary. In section 5, we formalize this subdivision by introducing the notation of bauble theory.

Another new tool is a topologically rigidity result, Proposition 8.1, which asserts that a homotopy equivalence from a pared manifold (M, P) to a manifold pair (N, Q), which is a homeomorphism on P and embeds a compact core for the components of $\partial M - P$ which are not pairs of pants into $\partial N - Q$ is homotopic to a pared homeomorphism.

In Lemma 4.2 we show that the set of minimally parabolic representations is dense in $\partial GF_0(M, P)$. Therefore, it suffices to show that any minimally parabolic representation in $\partial GF_0(M, P)$ may be approximated by geometrically finite representations. Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be an allowable collection of elements and let $d_{\mathcal{A}}$ be the associated metric on $AH(\pi_1(M), \pi_1(P))$.

Let $[\bar{\rho}]$ be a minimally parabolic representation in $\partial GF_0(M, P)$. An observation of Bers together with Sullivan's rigidity theorem and Lemma 2.7 allows us to to produce a sequence $\{[\rho_n]\}$ in $GF_0(M, P)$ converging to $[\bar{\rho}]$ such that there is a sequence of pants decomposition of the degenerating portion of $\partial_c N_{\rho_n}$ whose lengths converge to 0. Proposition 6.1 shows that these pants decompositions are pinchable for all large enough n. See Lemma 7.1 for a precise statement.

In Theorem 9.1 we apply the local estimate (Theorem 2.8) to each test element to show that given a representation $[\rho] \in GF_0(M, P)$ and a pinchable pants decomposition C of a collection R of components of $\partial_c N_\rho$, there exists a geometrically finite representation $[\widehat{\rho}]$ within $O(L)$ of $[\rho]$ (where L is the total length of C) such that $N_{\widehat{\rho}} \cup \partial_c N_{\widehat{\rho}}$ is homeomorphic to $N_\rho \cup (\partial N_\rho - C)$. In brief, Lemma 2.7 produces an infinite path $\beta\colon [0, \infty) \to \mathcal{T}(\partial M)$ which begins at a lift of ρ to $\mathcal{T}(\partial M - P)$ and which pinches the length of the pants decomposition to 0. For each t the tangent vector $\beta'(t)$ is represented by a unit norm Beltrami differential supported on the thin part. The local estimate, Theorem 2.8, gives that $\Upsilon_{a_i}(\beta)$ has length $O(L)$ in $CC_0(S^1 \times D^2)$ for each $a_i \in \mathcal{A}$ which is not conjugate to an element of $\pi_1(P)$. It is then easily checked that $q_M(\beta)$ has length $O(L)$ in the $d_{\mathcal{A}}$-metric on $AH(\pi_1(M), \pi_1(P))$ and hence accumulates at some conjugacy class $[\widehat{\rho}]$ in $\partial GF_0(M, P)$. We then check that $\partial_c N_\rho - R$ conformally embeds in $\partial_c N_{\widehat{\rho}}$ and that all components of C have become cusps. One may then apply our topological rigidity result, Proposition 8.1, and the fact that all ends associated to thrice punctured sphere components are geometrically finite, to check that $N_{\widehat{\rho}} \cup \partial_c N_{\widehat{\rho}}$ is homeomorphic to $N_\rho \cup (\partial N_\rho - C)$ and hence that $\widehat{\rho}$ is geometrically finite. (In [17], the fact that $\widehat{\rho}$ was geometrically finite was a direct consequence of a result of Keen-Maskit-Series [27] since C was a pants decomposition of the entire conformal boundary.)

In Section 10, we apply Theorem 9.1 to each $[\rho_n]$ producing a sequence of geometrically finite representation $\{[\widehat{\rho}_n]\}$ in $\partial GF_0(M,P)$ which also converges to $[\bar{\rho}]$. We also prove Corollary A in this final section.

Remark. In this remark we indicate how one may replace the use of Theorem 2.8 with an application of Brock, Canary and Minsky's recent resolution of the Ending Lamination Conjecture for hyperbolic 3-manifolds with relatively incompressible boundary to give a different proof of our main theorem in this case. We should note that our Main Theorem does not follow immediately from the Ending Lamination Theorem, even in the case that $\partial M - P$ is incompressible, since the ending invariants vary discontinuously over $AH(\pi_1(M), \pi_1(P))$ (see Brock [12] or Anderson-Canary-McCullough [3].) Moreover, there are substantive technical details involved in making the outline below into a complete argument.

Let (M,P) be a pared 3-manifold such that $\partial M - P$ is incompressible. Again, using Lemma 4.2, it suffices to show that minimally parabolic representations in the boundary of $GF_0(M,P)$ may be approximated by geometrically finite representations in $\partial GF_0(M,P)$. Let $\bar{\rho}$ be a minimally parabolic representation in $\partial GF_0(M,P)$ and let $\{\rho_n\}$ and $\{C_n\}$, be the sequences of representations in $GF_0(M,P)$ and pinchable collections of curves on $\partial_c N_{\rho_n}$ produced using Lemma 2.7 and Proposition 6.1 as indicated above. For all n, let $h_n : M - P \to N_{\rho_n} \cup \partial_c N_{\rho_n}$ be a homeomorphism in the homotopy class determined by ρ. One may then use Thurston's geometrization theorem and the quasiconformal deformation theory of geometrically finite Kleinian groups to produce a sequence of geometrically finite representations $\{[\widehat{\rho}_n]\}$ in $\partial GF_0(M,P)$ such that $N_{\widehat{\rho}_n} \cup \partial_c N_{\widehat{\rho}_n}$ is homeomorphic to $M - (P \cup h_n^{-1}(C))$ and $\partial_c N_{\bar{\rho}}$ is conformally identified with a subset of $\partial_c N_{\widehat{\rho}_n}$. One may apply convergence results of Thurston ([45, 46]) to show that $\{[\widehat{\rho}_n]\}$ converges in $AH(\pi_1(M), \pi_1(P))$ to some representation $[\bar{\rho}']$. One then shows that the conformal boundary of $\partial_c N_{\bar{\rho}'}$ contains a subsurface conformal to $\partial_c N_{\bar{\rho}}$ (in a manner similar to the technique used in the proof of Theorem 9.1) and applies the continuity of Thurston's length function (see Brock [11]) to verify that the ending laminations of the geometrically infinite ends of $(N_{\bar{\rho}'})^0_\epsilon$ are the same as those of $(N_{\bar{\rho}})^0_\epsilon$. The Ending Lamination Theorem may then be used to check that $\bar{\rho}' = \bar{\rho}$.

4. The boundary of $GF_0(M,P)$**.** In this section, we show that if a pared manifold is not maximal, then the boundary of $GF_0(M,P)$ is nonempty and contains a dense set of minimally parabolic representations. (Notice that if (M,P) is a maximal pared manifold then $GF_0(M,P)$ contains exactly one point, see Keen-

Maskit-Series [27], so $GF_0(M, P)$ has no boundary.) The proof that the boundary of $GF_0(M, P)$ is nonempty was provided by Peter Shalen.

LEMMA 4.1. *If a pared manifold* (M, P) *is not maximal, then the boundary of* $GF_0(M, P)$ *is nonempty.*

Proof of Lemma 4.1. Let Y be the component of $X(\pi_1(M), \pi_1(P))$ which contains $GF_0(M, P)$. Since (M, P) is not maximal, it has complex dimension at least 1. Let $g \in \pi_1(M)$ be an element such that $\bar{\tau}_g$ is nonconstant on Y. (Such an element must exist by Lemma 2.1.) The map $\tau_g : \mathcal{R}(\pi_1(M), \pi_1(P)) \to \mathbf{C}$ is algebraic, so the image of each component of $\mathcal{R}(\pi_1(M), \pi_1(P))$ is either a single point or the complement of finitely many points. It follows that $\bar{\tau}_g$ takes on all but finitely many values on Y. Hence, there exists $[\rho] \in Y$ such that $\bar{\tau}_g(\rho)$ is real and lies in $[0, 4)$, which implies that $\rho(g)$ is elliptic. It follows that $[\rho]$ does not lie in $GF_0(M, P)$. Since $GF_0(M, P)$ is an open subset of Y, it must therefore have nonempty boundary in Y, and hence in $X(\pi_1(M), \pi_1(P))$. □

The proof that minimally parabolic representations are dense in $\partial GF_0(M, P)$ is the natural generalization of the proof of the first part of Lemma 15.2 in [17].

LEMMA 4.2. *Let* (M, P) *be a pared 3-manifold. Then minimally parabolic representations are dense in* $\partial GF_0(M, P)$.

Proof of Lemma 4.2. Recall that Sullivan [43] proved that $GF_0(M, P)$ is the interior of its closure in $AH(\pi_1(M), \pi_1(P))$ and Kapovich (Theorem 8.44 in [26]) proved that every point in $AH(\pi_1(M), \pi_1(P))$ is a smooth point of $X = X(\pi_1(M), \pi_1(P))$. Let $Z \subset X$ be the set of all representations in X which are not minimally parabolic. Note that Z is a countable union of complex algebraic subsets of X and that Z does not intersect $GF_0(M, P)$.

If $[\rho_0] \in \partial GF_0(M, P)$ is not a limit of minimally parabolic representations in $\partial GF_0(M, P)$, then $[\rho_0]$ has a smooth, connected open neighborhood U such that $U \cap \partial GF_0(M, P) \subset Z$. Let X_0 denotes the irreducible component of X which contains $[\rho_0]$. Then $Z_0 = Z \cap X_0$ is a countable union of proper complex algebraic subvarieties of X_0. Since $U \subset X_0$ is smooth and connected, $W = U - (U \cap Z)$ is a connected, dense subset of U. Since W is connected and meets $GF_0(M, P)$ but is disjoint from $\partial GF_0(M, P)$, it must be that $W \subset GF_0(M, P)$. Hence U is contained in the closure of $GF_0(M, P)$ in X. But, since $GF_0(M, P)$ is the interior of its closure, this implies that $[\rho_0]$ lies in the interior of $GF_0(M, P)$, which is a contradiction. □

5. Convergence results and bauble theory. In this section, we produce a subdivision of the domains of discontinuity of Kleinian groups approximating a minimally parabolic representation into "surviving" components and "degenerating" components. The key tool is a result of Evans [20] which guarantees that the domains of discontinuity of the approximates converge to the domain of discontinuity of the limit.

We recall that a sequence $\{O_n\}$ of open sets in $\widehat{\mathbf{C}}$ converges, in the sense of Carathéodory, to an open set O if and only if:

(1) If K is a compact subset of O, then K lies in O_n for all sufficiently large n, and

(2) if U is open subset of O_n for infinitely many n, then $U \subset O$.

THEOREM 5.1. (Evans [20]) *Suppose that $\rho \in \mathcal{D}(\pi_1(M), \pi_1(P))$ is minimally parabolic, and $\{\rho_n\}$ is a sequence in $\mathcal{D}(\pi_1(M), \pi_1(P))$ which converges to ρ, then $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$ in the sense of Carathéodory.*

Remark. Evans actually states this result in the case that $\Omega(\rho)$ is nonempty, in which case he uses it to show that the convergence is actually strong. The result is obvious in the case that $\Omega(\rho)$ is empty, since, in general, $\Omega(\rho)$ contains the Caratheodory limit of any subsequence of $\{\Omega(\rho_n)\}$.

We will need a refinement of Evans' result, which shows that, given a component Δ of $\Omega(\rho)$, one may choose components of $\Omega(\rho_n)$ which converge to Δ. Notice that Δ need not be simply connected.

LEMMA 5.2. *Suppose that $\rho \in \mathcal{D}(\pi_1(M), \pi_1(P))$ is minimally parabolic and $\{\rho_n\}$ is a sequence in $\mathcal{D}(\pi_1(M), \pi_1(P))$ which converges to ρ. Let Δ be a component of $\Omega(\rho)$ and let $z \in \Delta$. Then*

(1) *there exists, for all sufficiently large n, a component Δ_n of $\Omega(\rho_n)$ which contains z, and*

(2) *$\{\Delta_n\}$ converges to Δ in the sense of Carathéodory,*

Proof of Lemma 5.2. The first part of the statement follows immediately from the fact that $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$. Moreover, there exists $\delta > 0$, such that, for all sufficiently large n, $\Omega(\rho_n)$, and hence Δ_n, contains the closed ball of radius δ about z. Therefore, any subsequence of $\{\Delta_n\}$ has a convergent subsequence. So, it suffices to show that if a subsequence of $\{\Delta_n\}$ converges to an open subset U of $\widehat{\mathbf{C}}$, then $U = \Delta$.

Since $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$ it is clear that $U \subset \Omega(\rho)$. Let y be any point in Δ. There exists a path $\gamma \in \Delta$ joining z to y. Since $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$, γ lies in $\Omega(\rho_n)$ for all sufficiently large n, so $y \in \Delta_n$ for all sufficiently large n. Therefore, $\Delta \subset U$.

It remains to show that $U \subset \Delta$. Let Θ be the stabilizer of Δ in $\rho(\pi_1(M))$. Then Θ is finitely generated and Δ is a component of $\Omega(\Theta)$ (see Lemma 2 of Ahlfors [2]). Moreover, see Theorem 3 in Maskit [30], this implies that every other component Δ' of $\Omega(\Theta)$ is a Jordan domain whose stabilizer in Θ is a quasifuchsian subgroup whose limit set is $\partial\Delta'$. (Recall that a finitely generated Kleinian group

R is *quasifuchsian* if $\Lambda(R)$ is a Jordan curve and each element of R preserves each component of $\Omega(R)$.) If U does not equal Δ, then U contains a point y in another component of $\Omega(\rho)$ and hence in another component Δ' of $\Omega(\Theta)$. Let Q be the quasifuchsian stabilizer of Δ' in Θ. Then, Marden's Stability Theorem (Proposition 9.1 in [29]) implies that, for all large enough n, $Q_n = \rho_n(\rho^{-1}(Q))$ is a quasifuchsian subgroup of $\rho_n(\pi_1(M))$ such that $\Lambda(Q_n)$ separates z from y. Therefore, for all large n, y does not lie in Δ_n which contradicts the fact that $y \in U$. Thus, it must be the case that $U = \Delta$. □

We can now divide up the domain of discontinuity of $\rho_n(\pi_1(M))$, for all large enough n, into those components which are converging to components of $\Omega(\rho)$ and those components which are "degenerating" in the limit.

Suppose that $\rho \in \mathcal{D}(\pi_1(M), \pi_1(P))$ is minimally parabolic, and $\{\rho_n\}$ is a sequence in $\mathcal{D}(\pi_1(M), \pi_1(P))$ which converges to ρ. Let $\{\Delta^1, \ldots, \Delta^m\}$ be a maximal collection of nonconjugate components of $\Omega(\rho)$ and let $\{z^1, \ldots, z^m\}$ be a collection of points with $z^s \in \Delta^s$ for all s. We call the pair $(\{\Delta^1, \ldots, \Delta^m\}, \{z^1, \ldots, z^m\})$ a *bauble*. By Lemma 5.2, there exists, for all s and sufficiently large n, a component Δ^s_n of $\Omega(\rho_n)$ which contains z^s. When Δ^s_n exists for all s, we call $(\{\Delta^1_n, \ldots, \Delta^m_n\}, \{z^1, \ldots, z^m\})$ an *approximate bauble* for ρ_n.

If $(\{\Delta^1_n, \ldots, \Delta^m_n\}, \{z^1, \ldots, z^m\})$ is an approximate bauble for ρ_n, then let $\Omega_F(\rho_n)$ denote the components of $\Omega(\rho_n)$ which are translates of Δ^s_n for some s. If Δ is a component of $\Omega_F(\rho_n)$, then there exists $g \in \pi_1(M))$ and s such that $\Delta = \rho_n(g)(\Delta^s_n)$ and Δ lies naturally in the sequence $\{\rho_n(g)(\Delta^s_n)\}$ which converges to $\rho(g)(\Delta^s)$. Let $\Omega_D(\rho_n) = \Omega(\rho_n) - \Omega_F(\rho_n)$. Notice that $\Omega_D(\rho_n)$ and $\Omega_F(\rho_n)$ are only well defined for large enough n and depend on the initial choice of bauble. It is then natural to divide $\partial_c N_{\rho_n}$ into $F_n = \Omega_F(\rho_n)/\rho_n(\pi_1(M))$ and $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$. Although the definition of D_n and F_n depend on the choice of bauble, we will not make this explicit in the notation. It will be important to note that if $\{[\rho_n]\}$ is a sequence in $GF_0(M, P)$, then, for all sufficiently large values of n, D_n is nonempty and no component of D_n is a thrice-punctured sphere.

LEMMA 5.3. *Suppose that $[\bar{\rho}] \in \partial GF_0(M, P)$ is minimally parabolic, and $\{[\rho_n]\}$ is a sequence in $GF_0(M, P)$ such that $\{\rho_n\}$ converges to $\bar{\rho}$. If $(\{\Delta^1, \ldots, \Delta^m\}, \{z^1, \ldots, z^s\})$ is a bauble for $\bar{\rho}$, then, for all sufficiently large values of n, $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$ is nonempty and no component of D_n is a thrice-punctured sphere.*

Proof of Lemma 5.3. If $\bar{\rho}$ were geometrically finite, then Marden's Stability Theorem (Proposition 9.1 in [29]) would imply that $\bar{\rho} \in GF_0(M, P)$. Therefore, $\bar{\rho}$ is geometrically infinite. If $(\{\Delta^1, \ldots, \Delta^m\}, \{z^1, \ldots, z^m\})$ is a bauble for ρ. Then, by Lemma 5.2, there exists an approximating bauble $(\{\Delta^1_n, \ldots, \Delta^m_n\}, \{z^1_n, \ldots, z^m_n\})$ for ρ_n for all sufficiently large values of n. For each n, let $h_n : \partial M - P \to N \cup \partial_c N_{\rho_n}$ be a homeomorphism in the homotopy class determined by ρ_n.

Fix ϵ such that $0 < \epsilon < \mu$. Let $(\bar{M}, \bar{P})$ be a relative compact core for $(N_{\bar{\rho}})^0_\epsilon$. Since $\bar{\rho}$ is geometrically infinite, there exists a geometrically infinite end of $(N_{\bar{\rho}})^0_\epsilon$. If $\Omega(\rho)$ is empty, then $D_n = \partial_c N_\rho$. If $\Omega(\rho)$ is nonempty, then Evans, Corollary 7.2 in [21], showed that there exists, for all sufficiently large n, a relative compact core (M_n, P_n) for $(N_{\rho_n})^0_\epsilon$ and a pared homeomorphism $j_n : (M_n, P_n) \to (\bar{M}, \bar{P})$ in the homotopy class determined by $\bar{\rho} \circ \rho_n^{-1}$. (Moreover, he shows that N_ρ is topologically tame under these same assumptions.) Since ρ_n is geometrically finite, this implies that there are at least $m+1$ components of $\partial_c N_{\rho_n}$. Since there are at most m components of $F_n = \partial_c N_{\rho_n} - D_n$, there must be at least one component of D_n for all sufficiently large n.

Suppose that D_n contains a thrice-punctured sphere component for infinitely many values of n. Then there exists a thrice-punctured sphere component Z of $\partial M - P$ such that $h_n(Z)$ lies in D_n for infinitely many values of n. We pass to a subsequence, again called $\{\rho_n\}$, such that $h_n(Z) \subset D_n$ for all n. Let H be a subgroup of $\pi_1(M)$ conjugate to $\pi_1(Z)$. Then, by Lemma 2.4, $\rho_n(H)$ is Fuchsian for all n and $\bar{\rho}(H)$ is Fuchsian. Let Δ_n be the component of $\Omega(\rho_n(H))$ which lies in $\Omega_D(\rho_n)$. Then, $\{\Delta_n\}$ has a subsequence converging to a component Δ of $\Omega(\bar{\rho}(H))$. Since $\{\Omega(\rho_n)\}$ converges to $\Omega(\bar{\rho})$, Δ lies in $\Omega(\bar{\rho})$. Therefore, $\Delta_n \subset \Omega_F(\rho_n)$ for infinitely many values of n, contradicting our assumption that $\Delta_n \subset \Omega_D(\rho_n)$ for all n. Therefore, for all sufficiently large values of n, D_n does not contain any thrice-punctured sphere components. □

6. Finding pinchable collections. In this section we observe that bounded length pants decompositions C_n of the degenerating portion D_n of $\partial_c N_{\rho_n}$ are eventually pinchable. Essentially, we must show that the curves in C_n are associated to distinct primitive hyperbolic elements of $\rho_n(\pi_1(M))$. The proof is a generalization of the proof of Proposition 3.1 in [17].

PROPOSITION 6.1. *Suppose that* $[\bar{\rho}] \in \partial GF_0(M, P)$ *is minimally parabolic,* $K > 0$*, and* $(\{\Delta^1, \ldots, \Delta^m\}, \{z^1, \ldots, z^m\})$ *is a bauble for* $\bar{\rho}$*. If* $\{[\rho_n]\}$ *is a sequence in* $GF_0(M, P)$*, such that* $\{\rho_n\}$ *converges to* ρ*, and, for all large enough* n*,* C_n *is a collection of disjoint simple closed geodesics on* $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$ *of total length at most* K*, then* C_n *is pinchable for all sufficiently large* n*.*

Proof of Proposition 6.1. We first pick x_0 in the interior of $C(N_{\bar{\rho}})$. Let $\widetilde{x}_0$ be a point in $CH(\Lambda(\bar{\rho}(\pi_1(M))))$ which projects to x_0. Theorem 5.1 implies that $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$, so $\widetilde{x}_0$ lies in the interior of $CH(\Lambda(\rho_n(\pi_1(M))))$ for all sufficiently large n. Let $\{g_1, \ldots, g_k\}$ be a set of generators for $\pi_1(M)$. Since $\{\rho_n\}$ converges, there exists $S > 0$ such that $d(\widetilde{x}_0, \rho_n(g_i)(\widetilde{x}_0)) \le S$ for all i and n, and there exists $\delta > 0$ such that $d(\widetilde{x}_0, \widetilde{x}) \le S$ implies that $d(\widetilde{x}, \gamma(\widetilde{x})) \ge 2\delta$ for all $\gamma \in \rho_n(\pi_1(M) - \{id\})$ and any n. Therefore, if $\gamma_{i,n}$ is a geodesic loop in $N_n = N_{\rho_n}$ based at x_0 representing $\rho_n(g_i)$, then $\gamma_{i,n}$ has length at most S and $\mathrm{inj}_{N_n}(x) \ge \delta$ at any point x of $\gamma_{i,n}$. (Here we have implicitly identified $\pi_1(N_n, x_0)$ with $\rho_n(\pi_1(M))$.)

Let $\bar{r}_n : N_n \cup \partial_c N_n \to \partial C(N_n)$ be the nearest point retraction. Theorem 2.2 implies that $\bar{r}_n(C_n)$ is homotopic, in $\partial C(N)$, to a collection C_n' of curves in $\partial C(N)$ of length at most $K' = 45Ke^{K/2}$. For each n, choose $\epsilon_n > 0$, such that:

(1) C_n' is contained in $C(N_n)_{\epsilon_n}$,

(2) $\epsilon_n < \mu$ where μ is the Margulis constant,

(3) $e^{K''}\sinh(2\epsilon_n) < \delta$ where $K'' = \sqrt{\frac{K'+1}{\pi}} + \frac{2K'+1}{\delta} + 1$, and

(4) any curve in $P(N_n)_{\epsilon_n}$ which is homotopic into C_n' is homotopic to a curve of length at most 1 on $P(N_n)_{\epsilon_n}$.

Let $h_n : M - P \to N_n \cup \partial_c N_n$ be a homeomorphism in the homotopy class determined by ρ and let $j_n : N_n \cup \partial_c N_n \to C(N_n)$ be a homeomorphism properly homotopic to $\bar{r}_n$. Let $k_n : C(N_n) \to C(N_n)_{\epsilon_n} - P(N_n)_{\epsilon_n}$ be a homeomorphism which is the identity on C_n' (We may construct k_n, since $C(N_n) - C(N_n)_{\epsilon_n}$ is homeomorphic to $P(N_n)_{\epsilon_n} \times (0, \infty)$.) Proposition 2.12.4 in Canary-McCullough [18] implies that there exists a pared homeomorphism $\psi_n : (M, P) \to (C(N_n)_{\epsilon_n}, P(N_n)_{\epsilon_n})$ which agrees with $k_n \circ j_n \circ h_n$ off of a regular neighborhood of P. In particular, $\psi_n(h_n^{-1}(C_n))$ is homotopic to $\bar{r}_n(C_n)$ within $\partial C(N)$.

If the theorem fails, we can pass to a subsequence, again called $\{\rho_n\}$, such that C_n is not pinchable for any n. Therefore, there exists, for all n, a surface B_n in M which is either a compressing disk or an immersed essential annulus with boundary contained in $h_n^{-1}(C_n) \cup P$. Let $B_n' = \psi_n(B_n)$. After a proper homotopy, we may assume that B_n' is either a compressing disk with boundary in C_n' or an immersed essential annulus with boundary in $C_n' \cup P_n$ where $P_n = P(N_n)_{\epsilon_n}$. In particular, we may assume that the boundary of B_n' has length at most $2K' + 1$.

Claim. Each surface B_n' is homotopic, rel boundary, to a surface Y_n such that if $x \in Y_n$ and $\mathrm{inj}_{N_n}(x) \geq \delta$, then $d(x, C_n') \leq k(\delta)$ for some uniform constant $k(\delta)$.

We briefly describe the construction of Y_n from Proposition 3.1 of [17]. One first constructs a pleated disk or annulus X_n by subdividing $\partial B_n'$ into segments of length at most 1, extending this subdivision to a triangulation of B_n' with no internal vertices, and then pulling each simplex tight (relative to its vertices) so that each simplex is totally geodesic. Since the area of a hyperbolic triangle is at most the length of any of its sides X_n has area at most $2K' + 1$ The boundary ∂X_n of our pleated surface is homotopic to $\partial B_n'$ by a homotopy of track length at most one. We append this homotopy to X_n to form Y_n.

The proof of our claim in the cases in which B_n' is a compressing disk or an annulus with boundary in C_n' are handled exactly as in Proposition 3.1 of [17], in which case $k(\delta)$ may be chosen to be $\frac{2K'}{\delta} + \sqrt{\frac{K'}{\pi}} + 1$. So, we may assume that B_n' is an annulus with boundary in $P_n \cup C_n'$. If $x \in X_n$ and $\mathrm{inj}_{N_n}(x) \geq \delta$, then the argument in [17] gives that $d(x, \partial X_n) \leq \sqrt{\frac{K'+1}{\pi}} + \frac{2K'+1}{\delta}$. So, $d(x, \partial B_n') \leq K'' =$

$\sqrt{\frac{K'+1}{\pi}} + \frac{2K'+1}{\delta} + 1$. But if $d(x, P_n) \le K''$, then, by inequality 1 in Section 2.2,

$$inj_{N_n}(x) \le e^{K''} \sinh(2\epsilon_n) < \delta$$

which contradicts our assumption that $\mathrm{inj}_{N_n}(x) \ge \delta$. So, in this case, $d(x, C_n') \le K''$. Thus, we have completed the proof of our claim with $k(\delta) = K''$.

Since the surface Y_n is essential and the loops $\{\gamma_{i,n}\}$ represent generators of $\pi_1(N_n, x_0)$, at least one of these loops, say $\gamma_{j,n}$, must intersect Y_n. Since $\gamma_{j,n}$ has length at most S and every point on $\gamma_{j,n}$ has injectivity radius at least δ, there exists a point y_n' on C_n' which lies a distance of at most $\widehat{K} = S + k(\delta)$ from x_0.

We may assume, without loss of generality, that we are working in the ball model and that $\tilde{x}_0$ is at the origin. Let $\tilde{y}_n'$ be a lift of y_n' which lies within $\widehat{K}$ of $\tilde{x}_0$ and let z_n be a point in $r_n^{-1}(\tilde{y}_n')$. Notice that $\bar{r}_n^{-1}(C_n')$ lies in D_n, since C_n' is homotopic to $\bar{r}_n(C_n)$ and $C_n \subset D_n$, so z_n lies in $\Omega_D(\rho_n)$. Let H_n be the support plane to $CH(\Lambda(\rho_n))$ passing through $\tilde{y}_n'$ and perpendicular to the geodesic ray joining $\tilde{y}_n'$ to z_n. Then H_n bounds an open disk $A_n \subset S^2$ which is contained within $\Omega_D(\rho_n)$. Moreover, A_n contains an open disk of (spherical) radius at least ϵ about z_n, where $\epsilon > 0$ depends only on $\widehat{K}$. We may pass to a subsequence, still named $\{\rho_n\}$, so that z_n converges to a point $z \in \widehat{\mathbf{C}}$. Since $\{\Omega(\rho_n)\}$ converges to $\Omega(\rho)$, the open ball of radius ϵ about z lies in $\Omega(\rho)$, so $z \in \gamma(\Delta^s)$ for some $\gamma = \rho(g) \in \rho(\pi_1(M))$ and some s. Since, by Lemma 5.2, $\rho_n(g)(\Delta_n^s)$ converges to $\gamma(\Delta^s)$, we see that, for all sufficiently large n, $A_n \cap \rho_n(g)(\Delta_n^s)$ is nonempty, so $A_n \subset \rho_n(g)(\Delta_n^s) \subset \Omega_F(\rho_n)$. Since $\Omega_F(\rho_n)$ and $\Omega_D(\rho_n)$ are disjoint, by definition, this contradiction establishes the result. □

7. Finding short pinchable collections. We next see that if $[\bar{\rho}]$ is minimally parabolic and lies in $\partial GF_0(M, P)$, then we can find a sequence $[\rho_n] \in GF_0(M, P)$ which converges to $[\rho]$ such that there is a sequence $\{C_n\}$ of pinchable pants decompositions of the "degenerating portions" D_n of $\partial_c N_{\rho_n}$ such that $\{l(C_n)\}$ converges to 0. This is the analogue of Proposition 4.1 in [17]. The key tools in the proof are Lemma 2.7, Proposition 6.1 and Sullivan's rigidity theorem.

LEMMA 7.1. *Suppose that $\bar{\rho}$ is minimally parabolic, $(\{\Delta^1, \ldots, \Delta^m\}, \{z^1, \ldots, z^m\})$ is a bauble for $\bar{\rho}$, and $[\bar{\rho}] \in \partial GF_0(M, P)$. Then there exists a sequence $\{[\rho_n]\}$ in $GF_0(M, P)$ converging to $[\bar{\rho}]$ such that, for all n, there exists a pinchable pants decomposition C_n of $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$ and $\{l(C_n)\}$ converges to 0.*

Proof of Lemma 7.1. Let $\{[\rho_j']\}$ be a sequence in $GF_0(M, P)$ which converge to $[\bar{\rho}]$, such that $\{\rho_j'\}$ converges to $\bar{\rho}$. By eliminating finitely many terms, we may assume that an approximating bauble $(\{\Delta_j^1, \ldots, \Delta_j^m\}, \{z^1, \ldots, z^m\})$ exists for all ρ_j'. Let $D_j = \Omega_D(\rho_j')/\rho_j'(\pi_1(M))$. Bers' inequality (see [8]) implies that there exists a pants decomposition C_j of D_j such that $l(C_j) \le \kappa$ for some uniform constant κ depending only on the topology of $\partial M - P$.

Applying Lemma 2.7 we produce, for each j, a sequence $\{\rho_{j,n}\}$ in $GF_0(M,P)$ such that for each j and n there exists a K^n-quasiconformal map $\widetilde{\phi}_{j,n}$ conjugating ρ'_j to $\rho_{j,n}$ such that $\widetilde{\phi}_{j,n}$ is conformal on $\Omega_F(\rho'_j)$ and $l(\phi_{j,n}(C_j)) \leq \frac{\kappa}{2^n}$. We may normalize so that $\widetilde{\phi}_{j,n}(z_1) = z_1$ and $d\widetilde{\phi}_{j,n}|_{z_1}$ is the identity map. If we define $D_{j,n} = \phi_{j,n}(D_j)$, then $C_{j,n} = \phi_{j,n}(C_j)$ is a pants decomposition of $D_{j,n}$ of length at most $\frac{\kappa}{2^n}$.

For each fixed n, $\{\widetilde{\phi}_{j,n}\}$ is a sequence of K^n-quasiconformal maps such that $\widetilde{\phi}_{j,n}(z_1) = z_1$ and $d\widetilde{\phi}_{j,n}|_{z_1}$ is the identity map. Therefore, $\{\widetilde{\phi}_{j,n}\}$ is a normal family. If ψ is a limit of a convergent subsequence, then ψ is K^n-quasiconformal, ψ is conformal on $\Omega(\bar{\rho})$ (since $\{\Omega_F(\rho'_j)\}$ converges to $\Omega(\bar{\rho})$), $\psi(z_1) = z_1$, $d\psi|_{z_1}$ is the identity map, and $\bar{\rho}_n = \psi\bar{\rho}\psi^{-1} \in \mathcal{D}(\pi_1(M), \pi_1(P))$. Sullivan's rigidity theorem [42] implies that ψ is conformal and hence the identity map, so $\bar{\rho}_n = \bar{\rho}$ for all n. Thus, for each n, $\{\widetilde{\phi}_{j,n}\}$ converges to the identity map and $\{\rho_{j,n}\}$ converges to $\bar{\rho}$.

For each n we choose $j(n)$ such that $z_s \in \widetilde{\phi}_{j(n),n}(\Delta^s_{j(n)})$ for all s, and

$$d_R(\rho_{j(n),n}, \bar{\rho}) \leq \frac{1}{n}$$

(where d_R is some metric on $\mathcal{D}(\pi_1(M), \pi_1(P))$). Let $\rho_n = \rho_{j(n),n}$. It then follows that $C_n = C_{j(n),n}$ is a pants decomposition of $D_n = \phi_{j(n),n}(D_{j(n),n})$ of length at most $\frac{\kappa}{2^n}$. Notice that $\{\rho_n\}$ converges to $\bar{\rho}$ and that $(\{\widetilde{\phi}_{j(n),n}(\Delta^1_{j(n)}), \ldots, \widetilde{\phi}_{j(n),n}(\Delta^m_{j(n)})\}, \{z^1, \ldots, z^m\})$ is an approximating bauble for all n. Thus, $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$ and Proposition 6.1 implies that C_n is pinchable for all large enough n. We may then remove finitely many terms to obtain the desired sequence. ◻

8. A topological rigidity result for pared manifolds. In this section, we develop a topological tool which will allow us to recognize the topological type of the hyperbolic manifold obtained by pinching a pinchable collection of curves. This topological realization will also allow us to conclude that the manifold is geometrically finite.

We recall that it is a consequence of Johannson's Classification Theorem (see Theorem 2.5) that any pared homotopy equivalence between acylindrical pared manifolds is homotopic to a pared homeomorphism. Waldhausen's theorem [47] assures us that any homotopy equivalence between Haken manifolds which preserves the boundary is a homeomorphism. The following proposition may be viewed as a mixture of these two results.

PROPOSITION 8.1. *Let (M,P) be a pared manifold and let $(\widehat{M},\widehat{P})$ be a compact, irreducible 3-manifold pair. Suppose that $f : (M,P) \to (\widehat{M},\widehat{P})$ is a map of pairs such that:*

(1) *f is a homotopy equivalence,*

(2) *$f|_P : P \to \widehat{P}$ is a homeomorphism, and*

(3) *there exists a submanifold Z of $\partial M - P$ which contains a compact core for each component of $\partial M - P$ which is not a thrice-punctured sphere, such that f embeds Z in $\partial N - Q$.*

Then f is homotopic to a homeomorphism of pairs.

Proof of Proposition 8.1. The key tool in our proof is the following lemma:

LEMMA 8.2. *Let (M, P) be a pared manifold. Then there exists a maximal pared manifold (M, Q) such that P is a collection of components of Q.*

We will give the proof of Proposition 8.1 and then return to the proof of Lemma 8.2.

Let (M, Q) be a maximal pared manifold such that P is a collection of components of Q. Lemma 2.6 implies that (M, Q) is an acylindrical pared manifold. We may assume that $Q-P$ is contained in the interior of Z. Let $\widehat{Q} = \widehat{P} \cup f(Q-P)$.

We next check that $(\widehat{M}, \widehat{Q})$ has relatively incompressible boundary. Since (M, Q) has relatively incompressible boundary, Proposition 1.2 in Bonahon [9] (see also Lemma 5.2.1 in Canary-McCullough [18]) implies that if $\pi_1(M) = G * H$ is any nontrivial free decomposition of $\pi_1(M)$, then there exists an element of $\pi_1(Q)$ which is not conjugate into either G or H. If $(\widehat{M}, \widehat{Q})$ does not have relatively incompressible boundary, then Dehn's Lemma gives a compressing disk with boundary in $\partial\widehat{M} - \widehat{Q}$, and hence a nontrivial free decomposition $\pi_1(\widehat{M}) = \widehat{G} * \widehat{H}$ such that every element of $\pi_1(\widehat{Q})$ is conjugate to an element of either $\widehat{G}$ or $\widehat{H}$. However, f_* induces an isomorphism of $\pi_1(M)$ to $\pi_1(\widehat{M})$ which takes the conjugacy class of any element of $\pi_1(Q)$ to the conjugacy class of an element of $\pi_1(\widehat{Q})$, so we have achieved a contradiction.

Theorem 2.5 then implies that f is homotopic to a homeomorphism of pairs between (M, Q) and $(\widehat{M}, \widehat{Q})$ and hence to a homeomorphism of pairs between (M, P) and $(\widehat{M}, \widehat{P})$. □

We now return to the proof of Lemma 8.2

Proof of Lemma 8.2. We will first observe that if (M, P) is not maximal, then there exists a pared manifold (M, Q') such that P is a proper subcollection of the components of Q'. One may iteratively apply this observation to find a maximal pared manifold (M, Q) such that P is a collection of components of Q.

Since (M, P) is not maximal, Lemma 4.1 implies that $GF_0(M, P)$ is nonempty and Lemma 4.2 implies that there exists a minimally parabolic representation $[\bar{\rho}] \in \partial GF_0(M, P)$. Let $\{[\rho_n]\}$ be a sequence in $GF_0(M, P)$ such that $\{\rho_n\}$ converges to $\{\bar{\rho}\}$. Let $(\{\Delta^1, \dots, \Delta^m\}, \{z^1, \dots, z^m\})$ be a bauble for $\bar{\rho}$. Then, by Lemma 5.3, there exists, for all sufficiently large values of n, an approximating bauble $(\{\Delta_n^1, \dots, \Delta_n^m\}, \{z_n^1, \dots, z_n^m\})$ for ρ_n such that $D_n = \Omega_D(\rho_n)/\rho_n(\pi_1(M))$ is nonempty and no component of D_n is a thrice-punctured sphere.

Bers' inequality [8] implies there exists a pants decomposition C_n of D_n of length at most $\kappa > 0$ (where κ depends only on the topology of $\partial M - P$). Now, Proposition 6.1 implies that C_n is pinchable for all sufficiently large n. Hence, we may choose $Q' = P \cup \mathcal{N}(h_n^{-1}(C_n))$ where $h_n : \partial M - P \to N_{\rho_n}$ is a homeomorphism in the homotopy class determined by ρ, $\mathcal{N}(h_n^{-1}(C_n))$ is a regular neighborhood of $h_n^{-1}(C_n)$ in $\partial M - P$, and n is large enough that C_n is pinchable. $\square$

9. The main global estimate. In this section we establish our main global estimate, which asserts that if $[\rho] \in GF_0(M,P)$ and C is a "short" pinchable pants decomposition of a collection R of components of $\partial_c N_\rho$, then $[\rho]$ is "near" to a geometrically finite representation $[\widehat{\rho}] \in \partial GF_0(M,P)$ such that each nontrivial element of $\widehat{\rho}(\pi_1(C))$ is parabolic.

The proof of the existence of $\widehat{\rho}$ is almost exactly the same as the proof of Proposition 6.1 in [17]. The key tool in the proof of existence is the main local estimate, Theorem 2.8. However, there are new technical difficulties associated with proving that $\widehat{\rho}$ is geometrically finite. We overcome these using bauble theory, the relative compact core and our topological rigidity result, Proposition 8.1. Our use of the relative compact core in this manner may be viewed as a generalization of earlier work of Maskit-Swarup [33] and Keen-Maskit-Series [27].

THEOREM 9.1. *Let (M,P) be a pared 3-manifold such that $\partial M - P$ is nonempty and let $\mathcal{A} = \{a_1,\dots,a_m\}$ be an allowable collection of test elements in $\pi_1(M)$. Given d_0 and d_1 such that $d_1 > 4d_0 > 0$, there exists $L_1 > 0$ and $G > 0$ such that if $[\rho] \in GF_0(M,P)$ and*

(1) *C is a pinchable pants decomposition of a collection R of components of $\partial_c N_\rho$,*

(2) *C has length $L < L_1$,*

(3) *no element of $\mathcal{A}$ represents a curve in C, and*

(4) *If a_i does not represent a curve in P, then $\frac{d_1}{2} \geq l_\rho(a_i) \geq 2d_0$ where $l_\rho(a_i)$ denotes the real translation distance of $\rho(a_i)$,*

then there exists a (conjugacy class of a) geometrically finite representation $[\widehat{\rho}] \in \partial GF_0(M,P)$ such that

$$d_{\mathcal{A}}([\rho],[\widehat{\rho}]) \leq G\,L.$$

Moreover, $[\widehat{\rho}] \in GF_0(M, P \cup \mathcal{N}(h^{-1}(C))$ where $h : M - P \to \partial_c N_\rho \cup N_\rho$ is a homeomorphism in the homotopy class determined by ρ and $\mathcal{N}(h^{-1}(C))$ is a regular neighborhood of $h^{-1}(C)$ in $\partial M - P$. In particular, if $R = \partial_c N_\rho$, then $\widehat{\rho}$ is a maximal cusp.

Proof of Theorem 9.1. Recall the quotient map $q_T : \mathcal{T}(T^2) \to CC_0(S^1 \times D^2)$ and identify $\pi_1(S^1 \times D^2)$ with $\mathbf{Z}$. Let

$$E_s = \left\{\sigma \in \mathcal{T}(T^2) \mid\, 2d_0 \leq l(q_T(\sigma)(1)) \leq d_1/2\right\}$$

and

$$E_f = \left\{\sigma \in \mathcal{T}(T^2) \mid d_0 \leq l(q_T(\sigma)(1)) \leq d_1\right\}$$

where $l(q_T(\sigma)(1))$ denotes the real translation distance of $q_T(\sigma)(1)$.

We may reorder $\mathcal{A}$ so that a_i is conjugate into $\pi_1(P)$ if and only $i > m_0$. (It is possible that $m_0 = m$.) By condition (4), $\Upsilon_{a_i}([\rho]) \in q_T(E_s)$ for all $i \leq m_0$. The sets $q_T(E_s)$ and $q_T(E_f)$ are compact subsets of $CC_0(S^1 \times D^2)$. Let δ be the (nonzero) distance, in the Teichmüller metric on $CC_0(S^1 \times D^2)$, between $q_T(E_s)$ and the boundary of $q_T(E_f)$.

Let c_0 and K_0 be the constants associated to d_0 in the main local estimate, Theorem 2.8. Let B_1 be the value of B produced by Lemma 2.7 when $L_0 = c_0$ and let $B_2 = mB_1$. Assuming that $L \leq c_0$, we may iteratively apply Lemma 2.7 (reducing the length of C by a factor of 2 at each stage) to produce an infinite path $\beta : [0, \infty) \to \mathcal{T}(\partial_c N_\rho)$ where $\beta(t) = (X_t, g_t)$ such that, for all $t \in [nB_2, (n+1)B_2]$,

$$l_{X_t}(g_t(C)) \leq \frac{L}{2^n}$$

and there exists $\mu_t \in B(X_t)$ such that:

(1) $D\Phi_{q_M(\beta(t))}(\mu_t) = \beta'(t)$,

(2) $\|\mu_t\| \leq 1$, and

(3) μ_t is supported on the $\frac{L}{2^{n-1}}$-thin part of X_t associated to $g_t(C)$.

Assuming that $L_1 \leq c_0 e^{-d_1}$, Theorem 2.8 implies that if $i \leq m_0$, then

$$\|D\Upsilon_{a_i}(D\bar{\Phi}_{q_M(\beta(t))}(\mu_t))\| = \|D\widetilde{\Upsilon}_{a_i}(\beta'(t))\| \leq K_0 \frac{L}{2^n}$$

for all $t \in [nB_2, (n+1)B_2]$ such that $\widetilde{\Upsilon}_{a_i}(\beta(t)) \in E_f$. Therefore, by integrating the above estimate we see that $\widetilde{\Upsilon}_{a_i}(\beta([0,\infty))) \cap E_f$ has length at most

$$K_0 B_2 \left(L + \frac{L}{2} + \cdots + \frac{L}{2^n} + \cdots\right) = 2K_0 B_2 L.$$

If $L_1 < \frac{\delta}{2K_0B_2}$, this implies that the entire path lies in E_f. Let

$$L_1 = \min\left\{c_0 e^{-d_1}, \frac{\delta}{2K_0 B_2}\right\}.$$

If $[\nu] \in CC_0(S^1 \times D^2)$, let $\bar{\tau}_0([\nu])$ denote the square of the trace of $\nu(1)$. Then $\bar{\tau}_0 : CC_0(S^1 \times D^2) \to \mathbf{C}$ is smooth and $\bar{\tau}_a([\rho']) = \bar{\tau}_0(\Upsilon_a([\rho']))$ for all $[\rho'] \in GF_0(M, P)$ and all $a \in \pi_1(M)$ such that a is not conjugate into $\pi_1(P)$. Since $q_T(E_f)$ is compact and $\bar{\tau}_0$ is smooth, there exists $K_3 > 0$ such that if $\nu_1, \nu_2 \in q_T(E_f)$, then

$$|\bar{\tau}_0(\nu_1) - \bar{\tau}_0(\nu_2)| \leq K_3 d(\nu_1, \nu_2)$$

(where d denotes the Teichmüller metric on $CC_0(S^1 \times D^2)$). Thus, for all $i \leq m_0$,

$$\bar{\tau}_0(q_T(\widetilde{\Upsilon}_{a_i}(\beta([0,\infty)))) = \bar{\tau}_{a_i}(q_M(\beta([0,\infty)))$$

has length at most $2K_3K_0B_2L$ in $\mathbf{C}$. (Notice that if $i > m_0$, then $\tau_{a_i}(q_M(\beta([0,\infty)))$ is constant.) Therefore, $q_M(\beta([0,\infty)))$ has length at most $2mK_3K_0B_2L$ in the $d_{\mathcal{A}}$-metric on $AH(\pi_1(M), \pi_1(P))$. Thus, there is a conjugacy class $[\widehat{\rho}] \in AH(\pi_1(M), \pi_1(P))$ which is an accumulation point of $\{q_M(\beta(n))\}$. Let $\{[\rho_n]\}$ be a subsequence of $\{q_M(\beta(n))\}$ which converges to $[\widehat{\rho}]$ such that $\{\rho_n\}$ converges to $\widehat{\rho}$.

Let $G = 2mK_0K_3B_2$. Then,

$$d_{\mathcal{A}}([\widehat{\rho}], [\rho]) \leq GL.$$

It only remains to show that there exists a homeomorphism

$$\widehat{h} : M - (P \cup h^{-1}(C)) \to \partial_c N_{\widehat{\rho}} \cup N_{\widehat{\rho}}$$

in the homotopy class determined by $\widehat{\rho}$. Notice that this implies that $\widehat{\rho}$ is geometrically finite.

Let h be the homeomorphism $h : M - P \to N_\rho \cup \partial_c N_\rho$ from the statement of the Theorem such that $h_* = [\rho]$. Let $C' = h^{-1}(C)$. If η is a curve in C', then $l_{X_t}(g_t(h(\eta)))$ converges to 0. Let b be an element of $\pi_1(M)$, such that η is a representative of $\rho(b)$. Proposition 6.1 of Sugawa [41] implies that the complex translation length of $\rho_t(b)$ also converges to 0. Thus, $\widehat{\rho}(b)$ is parabolic. So, every element of $\widehat{\rho}(\rho^{-1}(\pi_1(C)))$ is parabolic.

Let $T = \partial_c N_\rho - R$ and let $\Omega_T(\rho)$ denote the components of $\Omega(\rho)$ which cover components of T. Notice that since g_t is conformal on T for all t, there exists, for all n, a quasiconformal map $\widetilde{\phi}_n$ conjugating ρ to ρ_n which is conformal on $\Omega_T(\rho)$.

Let $\{U_1, \ldots, U_q\}$ denote a maximal collection of nonconjugate components of $\Omega_T(\rho)$. Since the stabilizer of each component of $\Omega(\rho)$ is a nonelementary Kleinian group, we may, for each s, pick $g_s, h_s \in \pi_1(M)$ such that $\rho(g_s)$ and $\rho(h_s)$ both stabilize U_s, $< g_s, h_s >$ is a free subgroup of $\pi_1(M)$ of rank 2 and $\widehat{\rho}(g_s)$ and $\widehat{\rho}(h_s)$ are hyperbolic. Notice that $\widetilde{\phi}_n(U_s)$ misses the four fixed points of $\rho_n(g_s)$ and $\rho_n(h_s)$ for all i, and that these fixed points converge to the four fixed point of $\widehat{\rho}(g_s)$ and $\widehat{\rho}(h_s)$. Montel's theorem then implies that $\{\widetilde{\phi}_n|_{U_s}\}$ is a normal family for all s. Pick a subsequence of $\{\rho_n\}$, again denoted $\{\rho_n\}$, such that for each s, $\{\widetilde{\phi}_n|_{U_s}\}$ converges to a map $\widetilde{\psi}_s$ with domain U_s. Then, for each s, either $\widetilde{\psi}_s$ is conformal or its image is a point.

Suppose that, for some s, $\widetilde{\psi}_s$ is a constant map with image w_s. Pick $z_s \in U_s$, then $\{\widetilde{\phi}_n(\rho_n(g_s)(z_s))\} = \{\rho_n(g_s)(\widetilde{\phi}_n(z_s))\}$ must converge to $\widehat{\rho}(g_s)(w_s)$. Therefore, w_s must be a fixed point of $\widehat{\rho}(g_s)$. We may similarly conclude that w_s is a fixed point of $\widehat{\rho}(h_s)$. Since $\widehat{\rho}(g_s)$ and $\widehat{\rho}(h_s)$ are noncommuting hyperbolic elements contained in a discrete group, their fixed points must be distinct. Therefore, $\widetilde{\psi}_s$ must be a conformal map for all s.

Since $\Omega(\widehat{\rho})$ contains the limit of any subsequence of $\{\Omega(\rho_n)\}$, we see immediately that $\widetilde{\psi}_s(U_s) \subset \Omega(\widehat{\rho})$. Let Θ_s be the stabilizer $\mathrm{stab}_{\rho(\pi_1(M))}(U_s)$ of U_s in $\rho(\pi_1(M))$ and let $G_s = \rho^{-1}(\Theta_s)$. Then, for all n, $\widetilde{\phi}_n(U_s)$ is a component of $\Omega(\rho_n)$ and $\rho_n(G_s) = \mathrm{stab}_{\rho_n(\pi_1(M))}(\widetilde{\phi}(U_s))$. Since $\{\rho_n\}$ converges to $\widehat{\rho}$ and $\{\widetilde{\phi}_n|_{U_s}\}$ converges to $\widetilde{\psi}_s$, $\widehat{\rho}(G_s) \subset \mathit{stab}_{\widehat{\rho}(\pi_1(M))}(\widetilde{\psi}_s(U_s))$. On the other hand, if $\widehat{\rho}(g)(\widetilde{\psi}_s(U_s)) \cap \widetilde{\psi}_s(U_s)$ is nonempty, then $\rho_n(g)(\widetilde{\phi}_n(U_s)) \cap \widetilde{\phi}_n(U_s)$ is nonempty for all large enough n, which implies that $g \in G_s$. Therefore, $\widetilde{\psi}_s$ descends to a conformal embedding $\psi_s : U_s/\rho(\pi_1(M)) \to \partial_c\widehat{N}$ where $\widehat{N} = N_{\widehat{\rho}}$.

We may combine the maps $\{\psi_1, \ldots, \psi_m\}$ to construct a conformal embedding $\psi : T \to \partial_c\widehat{N}$. Let $S = h^{-1}(T)$ and let $C' = h^{-1}(C)$. Let S_0 be a compact core for S and let $T_0 = h(S_0)$. Let $\widehat{j} : \widehat{N} \cup \partial_c\widehat{N} \to C(\widehat{N})$ be a homeomorphism properly homotopic to the nearest point retraction. Choose ϵ such that $\mu > \epsilon > 0$ and $\widehat{j}(T_0) \subset C(\widehat{N})_\epsilon$. Let $(\widehat{M}, \widehat{P})$ be a relative compact core for $C(\widehat{N})_\epsilon$ whose intersection with $\widehat{j}(T)$ contains $j(T_0)$.

Let $f : M \to \widehat{M}$ be a homotopy equivalence in the homotopy class determined by $\widehat{\rho}$. Let $P' = P \cup \mathcal{N}(C')$ where $\mathcal{N}(C')$ is a regular neighborhood of C' in $\partial M - P$. If β is a core curve of an annular component P_0 of P', then $f(\beta)$ is homotopic into a rank one cusp of $\widehat{N}$ and hence into an annular component of $\widehat{P}$. Since $\pi_1(P_0)$ is a maximal abelian subgroup of $\pi_1(M)$, $f(\beta)$ is homotopic to a core curve of a component of $\widehat{Q}$. Similarly, if P_0 is a toroidal component of P, then $f(P_0)$ is homotopic into a rank 2 cusp of $\widehat{N}$, and since $\pi_1(P_0)$ is a maximal abelian subgroup of $\pi_1(M)$, $f(P_0)$ is homotopic to a toroidal component of $\widehat{P}$. Therefore, we may assume that $f|_{P'}$ is a homeomorphism whose image is a collection $\widehat{P}'$ of components of $\widehat{P}$.

If Z is a component of $\partial M - P'$ which is not a thrice punctured sphere, then Z is a component of T. Let Z_0 be a compact core for Z which is a component of S_0. Then, $f|_{Z_0}$ is homotopic to the embedding $\widehat{j} \circ \psi \circ h|_{Z_0}$. Therefore, we may assume, after homotopy, that $f|_{S_0} = \widehat{j} \circ \psi \circ h|_{S_0}$. Proposition 8.1 then implies that f is homotopic to a pared homeomorphism $g : (M, P') \to (\widehat{M}, \widehat{P}')$.

We next argue that $\widehat{P} = \widehat{P}'$. Notice that the toroidal components of both P and $\widehat{P}$ are in a one-to-one correspondence with the conjugacy classes of maximal rank two free abelian subgroups of $\pi_1(M)$ and $\pi_1(\widehat{M})$. Since f is a homotopy equivalence, every toroidal component of $\widehat{P}$ lies in $\widehat{P}'$. Let Y be a pants decomposition of T_0 and let R be a collection of core curves of all the annular components of P'. Then $Y \cup R$ is a pants decomposition of the nontoroidal components of ∂M, and $f(Y) \cup f(R)$ is a collection of disjoint, nonparallel simple closed curves contained in nontoroidal components of $\partial\widehat{M}$, none of which bounds a disk in $\partial\widehat{M}$.

Since M is homotopy equivalent to $\widehat{M}$, the Euler characteristic of $\partial\widehat{M}$ must agree with the the Euler characteristic of ∂M. Hence, a pants decomposition of the nontoroidal components of $\widehat{M}$ must have the same number of curves as a pants decomposition of the nontoroidal components of M. It follows that $f(Y) \cup f(R)$ must be a pants decomposition of the nontoroidal boundary components of $\partial\widehat{M}$. In particular, $\widehat{P}$ must equal $\widehat{P}'$ and each component of $f(\partial T_0)$ must be parallel to an annular component of $\widehat{P}$ in $\partial\widehat{M}$.

Recall that the ends of $\widehat{N}^0_\epsilon$ are in one-to-one correspondence with the components of $\partial\widehat{M} - \widehat{P}$. If Z is a component of $\partial\widehat{M} - \widehat{P}$ which is homeomorphic to a thrice-punctured sphere, then its associated end is geometrically finite by Lemma 2.3. If Z is not homeomorphic to a thrice-punctured sphere, then it contains a component of $\psi(T_0)$, so the end associated to Z must be geometrically finite, since the component of $\widehat{N}^0_\epsilon - \widehat{M}$ bounded by Z contains a component of $\widehat{N} - C(\widehat{N})$. Since all the ends of $\widehat{N}^0_\epsilon$ are geometrically finite, $\widehat{N}$ is itself geometrically finite.

Since $\widehat{N}$ is geometrically finite, we may assume that $(\widehat{M}, \widehat{P}) = (C(\widehat{N})_\epsilon, P(\widehat{N})_\epsilon)$. Also note that, since the closure of $C(\widehat{N}) - C(\widehat{N})_\epsilon$ may be identified with $P(\widehat{N})_\epsilon \times [0,\infty)$, there exists a homeomorphism $\widehat{k} : \widehat{N} \cup \partial_c\widehat{N} \to C(\widehat{N})_\epsilon - P(\widehat{N})_\epsilon$ which is homotopic to $\widehat{j}$. Then $\widehat{k}^{-1} \circ g$ gives a homeomorphism from $M - P'$ to $\widehat{N} \cup \partial_c\widehat{N}$ in the homotopy class determined by $\widehat{\rho}$. Thus, $\widehat{\rho} \in GF_0(M, P')$. □

Remarks. (1) If one only requires that $\widehat{\rho}$ be geometrically finite, and does not insist on verifying the homeomorphism type, one may make a simpler argument which is more directly in the spirit of Keen-Maskit-Series [27]. While establishing that $\widehat{P} = \widehat{P}'$ we observed that $f(Y) \cup f(R)$ is a pants decomposition of the nontoroidal components of $\partial\widehat{M}$. This argument did not use Theorem 8.1 or the topological theory of pared manifolds. It then follows that every component of $\partial\widehat{M} - \widehat{P}$ which does not contain a component of $f(T_0)$ is a thrice punctured sphere, and hence that every end of $\widehat{N}^0_\epsilon$ is geometrically finite (as in the next to last paragraph of the proof of Theorem 9.1.)

(2) Bromberg [14] has recently given a new proof of Theorem 9.1, in the case that P has no annular components, using cone manifold techniques. His proof allows C to be any pinchable collection of curves.

10. Compiling the proof. The proof of our Main Theorem follows from Theorem 10.1 which asserts that minimally parabolic representations in $\partial GF_0(M, P)$ may be approximated by geometrically finite points in $\partial GF_0(M, P)$ and Lemma 4.2 which asserts that minimally parabolic representations are dense in $\partial GF_0(M, P)$. Notice that Theorem 10.1 generalizes Theorem 16.2 and the Main Theorem of [17].

THEOREM 10.1. *Let (M, P) be any pared 3-manifold. If $[\bar{\rho}] \in \partial GF_0(M, P)$ is minimally parabolic, then $[\bar{\rho}]$ may be approximated by (conjugacy classes of) geo-*

metrically finite representations in $\partial GF_0(M,P)$. If, in addition, $\Omega(\bar\rho) = \emptyset$, then $[\bar\rho]$ may be approximated by maximal cusps in $\partial GF_0(M,P)$.

Proof of Theorem 10.1. Let $(\{\Delta^1,\dots,\Delta^m\},\{z^1,\dots,z^n\})$ be a bauble for $\bar\rho$. Lemma 7.1 supplies a sequence $\{[\rho_n]\}$ in $GF_0(M,P)$ converging to $[\bar\rho]$ and a sequence $\{C_n\}$ of pinchable pants decompositions of $\{D_n\}$ whose lengths $\{l(C_n)\}$ converge to 0. Let $\mathcal{A} = \{a_1,\dots,a_m\}$ be an allowable collection of test elements, provided by Proposition 2.1.

We may reorder $\mathcal{A}$ so that a_i is conjugate into $\pi_1(P)$ if and only if $i > m_0$. We may choose positive constants d_0 and d_1 such that

$$4d_0 < l_\rho(a_i) < \frac{d_1}{4}$$

for all $i \le m_0$. Let L_1 be the constant provided by Proposition 9.1 with our chosen values of d_0 and d_1.

Since $\{[\rho_n]\}$ converges to $[\rho]$ and $\{l(C_n)\}$ converges to 0, there exists n_0 such that if $n \ge n_0$, then

(1) $2d_0 \le l_{\rho_n}(a_i) \le \frac{d_1}{2}$ for all $i \le m_0$,

(2) $45l(C_n)e^{l(C_n)/2} < d_0$, and

(3) $l(C_n) < L_1$.

If $\rho_n(c_n)$ is an element of $\rho_n(\pi_1(M))$ representing a curve in C_n and $n \ge n_0$, then Theorem 2.2 implies that there is a representative of $\rho_n(c_n)$ in $\partial C(N_{\rho_n})$ of length at most $45l(C_n)e^{l(C_n)/2} < d_0$. It follows that the real translation length $l(\rho_n(c_n))$ of $\rho_n(c_n)$ is less than d_0. Therefore, no curve in C_n is represented by an element of $\mathcal{A}$.

Theorem 9.1 implies that, for all $n > n_0$, there exists a geometrically finite point $[\widehat{\rho}_n] \in \partial GF_0(M,P)$ such that

$$d_{\mathcal{A}}([\rho_n],[\widehat{\rho}_n]) \le Gl(C_n).$$

Since, $\{d_{\mathcal{A}}([\rho_n],[\rho])\})$ and $\{l(C_n)\}$ both converge to 0, $\{[\widehat{\rho}_n]\}$ converges to $[\rho]$. Thus, $\{[\widehat{\rho}_n]\}$ is the desired sequence.

Notice that if $\Omega(\rho) = \emptyset$, then $D_n = \partial_c N_{\rho_n}$ and C_n is a pants decomposition of $\partial_c N_\rho$, so $\widehat{\rho}_n$ is a maximal cusp, for each n. □

Combining Lemma 4.2 and Theorem 10.1 gives our main result in the case that (M,P) is not maximal. If (M,P) is maximal then $GF_0(M,P) = AH(\pi_1(M), \pi_1(P))$ is a single point, see Keen-Maskit-Series [27], so our main result is vacuously true.

MAIN THEOREM. *Let (M,P) be a pared 3-manifold such that $\pi_1(M)$ is non-abelian and $\partial M - P$ is nonempty. Then conjugacy classes of geometrically finite representations are dense in the boundary of $GF_0(M,P)$.*

Corollary A, which generalizes Corollaries 15.4 and 16.4 in [17], follows nearly immediately from our main theorem.

COROLLARY A. *Let (M, P) be a pared 3-manifold with nonabelian fundamental group such that $\partial M - P$ is nonempty. Then the set of conjugacy classes $[\rho] \in \partial GF_0(M, P)$ such that N_ρ contains arbitrarily short geodesics is a dense G_δ subset of $\partial GF_0(M, P)$.*

Proof of Corollary A. If $g \in \pi_1(M)$, we consider the function

$$l_g : AH(\pi_1(M), \pi_1(P)) \to [0, \infty)$$

where $l_g([\rho])$ is the real translation length of $\rho(g)$. Since l_g is continuous for all g, the set $\mathcal{U}_n$ of representations in $\partial GF_0(M, P)$ whose associated manifolds contain a closed geodesic of length less than $\frac{1}{n}$ is open in $\partial GF_0(M, P)$ for all n. Since minimally parabolic representations are dense in $\partial GF_0(M, P)$ and geometrically finite points are dense in $\partial GF_0(M, P)$, $\mathcal{U}_n$ is dense in $\partial GF_0(M, P)$ for all n. The Baire category theorem then applies to show that $\bigcap_{n \in \mathbf{Z}_+} U_n$ is a dense G_δ in $\partial GF_0(M, P)$. $\square$

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REFERENCES

[1] W. Abikoff, The Euler characteristic and inequalities for Kleinian groups, *Proc. Amer. Math. Soc.* **97** (1986), 593–601.
[2] L. Ahlfors, The structure of a finitely generated Kleinian group, *Acta Math.* **122** (1969), 1–17.
[3] J. W. Anderson, R. D. Canary and D. McCullough, On the topology of deformation spaces of Kleinian groups, *Ann. of Math.* **152** (2000), 693–741.
[4] A. F. Beardon, *The Geometry of Discrete Groups, Grad. Texts in Math.*, vol. 91, Springer-Verlag, New York, 1983.
[5] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry, Universitext*, Springer-Verlag, New York, 1992.
[6] L. Bers, Simultaneous uniformization, *Bull. Amer. Math. Soc.* **66** (1960), 94–97.
[7] ———, On moduli of Kleinian groups, *Russian Math. Surveys* **29** (1974), 88–102.
[8] ———, An inequality for Riemann surfaces, *Differential Geometry and Complex Analysis*, Springer-Verlag, New York, 1985.
[9] F. Bonahon, Bouts des variétés hyperboliques de dimension 3, *Ann. of Math.* **124** (1986), 71–158.
[10] B. Bowditch, Geometrical finiteness for hyperbolic groups, *J. Funct. Anal.* **113** (1993), 245–317.
[11] J. Brock, Continuity of Thurston's length function, *Geom. Funct. Anal.* **10** (2000), 741–797.

[12] ———, Boundaries of Teichmüller spaces and end-invariants for hyperbolic 3-manifolds, *Duke Math. J.* **106** (2000), 527–552.
[13] J. Brock and K. Bromberg, On the density of geometrically finite hyperbolic 3-manifolds, *Acta Math.* (to appear).
[14] K. Bromberg, Hyperbolic cone-manifolds, short geodesics and Schwarzian derivatives, *J. Amer. Math. Soc.* (to appear).
[15] ———, Projective structures with degenerate holonomy and the Bers' density conjecture, preprint.
[16] R. D. Canary, The conformal boundary and the boundary of the convex core, *Duke J. Math.* **106** (2001), 193–207.
[17] R. D. Canary, M. Culler, S. Hersonsky and P. B. Shalen, Approximation by maximal cusps in boundaries of deformation spaces of Kleinian groups, *J. Differential Geom.* **64** (2003), 57–109.
[18] R. D. Canary and D. McCullough, Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups, *Mem. Amer. Math. Soc.* (to appear).
[19] D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces, *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, pp. 113–253.
[20] R. Evans, Weakly type-preserving sequences and stong convergence, *Geom. Dedicata* (to appear).
[21] ———, Tameness persists in weakly type-preserving strong limits, *Amer. J. Math.* **126** (2004), 713–737.
[22] F. P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, NY, 1987.
[23] Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller spaces,* Springer-Verlag, NY, 1992.
[24] W. Jaco, *Lectures on Three-manifold Topology, CBMS Regional Conf. Ser. Math.*, no. 43, Amer. Math. Soc., Providence, RI, 1977.
[25] K. Johannson, *Homotopy Equivalences of 3-manifolds, Lecture Notes in Math.*, vol. 761, Springer-Verlag, NY, 1979
[26] M. Kapovich, *Hyperbolic Manifolds and Discrete Groups: Lectures on Thurston's Hyperbolization*, Birkhäuser, Basel, 2000.
[27] L. Keen, B. Maskit and C. Series, Geometric finiteness and uniqueness of groups with circle packing limit sets, *J. Reine Angew. Math.* **436** (1993), 209–219.
[28] R. Kulkarni and P. Shalen, On Ahlfors' finiteness theorem, *Adv. in Math.* **76** (1989), 155–169.
[29] A. Marden, The geometry of finitely generated Kleinian groups, *Ann. of Math.* **99** (1974), 383–462.
[30] B. Maskit, On boundaries of Teichmüller spaces and on Kleinian groups: II, *Ann of Math.* **91** (1970), 607–639.
[31] ———, Self-maps on Kleinian groups, *Amer. J. Math.* **93** (1971), 840–856.
[32] ———, *Kleinian Groups*, Springer-Verlag, NY, 1988.
[33] B. Maskit and G. Swarup, Two parabolic generator Kleinian groups, *Israel J. Math.* **64** (1988), 257–266.
[34] D. McCullough, Compact submanifolds of 3-manifolds with boundary, *Quart. J. Math. Oxford* **37** (1986), 299–307.
[35] C. T. McMullen, Cusps are dense, *Ann. of Math.* **133** (1991), 217–247.
[36] Y. N. Minsky, The classification of punctured torus groups, *Ann. of Math.*, **149** (1999), 559–626.
[37] J. W. Morgan, On Thurston's uniformization theorem for three-dimensional manifolds, *The Smith Conjecture* (J. Morgan and H. Bass, eds.), Academic Press, Boston, 1984, pp. 37–125.
[38] J. W. Morgan and P. B. Shalen, Degeneration of hyperbolic structures, III: Actions of 3-manifold groups on trees and Thurston's compactness theorem, *Ann. of Math.* **127** (1988), 457–519.
[39] G. D. Mostow, Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms, *Publ. Inst. Hautes Études Sci. Math.* **34** (1968), 53–104.
[40] G. Prasad, Strong rigidity of Q-rank 1 lattices, *Invent. Math.* **21** (1973), 255–286.
[41] T. Sugawa, Uniform perfectness of the limit sets of Kleinian groups, *Trans. Amer. Math. Soc.* **353** (2001), 3603–3615.
[42] D. P. Sullivan, On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, *Riemann Surfaces and Related Topics, Ann. of Math. Stud.*, vol. 97, Princeton Univ. Press, Princeton, NJ, 1980, pp. 465–496.
[43] ———, Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity of Kleinian groups, *Acta Math.* **155** (1985), 243–260.
[44] W. P. Thurston, The Geometry and Topology of 3-manifolds, lecture notes, Princeton University.
[45] ———, Hyperbolic Structures on 3-manifolds, II: Surface Groups and 3-manifolds which fiber over the circle, available at http://front.math.ucdavis.edu.
[46] ———, Hyperbolic structures on 3-manifolds III: Deformations of 3-manifolds with incompressible boundary, available at http://front.math.ucdavis.edu.
[47] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, *Ann. of Math.* **87** (1968), 56–88.