THE TRIPLE INTERSECTION PROPERTY, THREE DIMENSIONAL EXTREMAL LENGTH, AND TILING OF A TOPOLOGICAL CUBE

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Dedicated to the memory of Oded Schramm

ABSTRACT. Let $\mathcal{T}$ be a triangulation of a closed topological cube Q, and let V be the set of vertices of $\mathcal{T}$. Further assume that the triangulation satisfies a technical condition which we call the *triple intersection property* (see Definition 3.6). Then there is an essentially unique tiling $\mathcal{C} = \{C_v : v \in V\}$ of a rectangular parallelepiped R by cubes, such that for every edge (u, v) of $\mathcal{T}$ the corresponding cubes C_v, C_u have nonempty intersection, and such that the vertices corresponding to the cubes at the corners of R are at the corners of Q. Moreover, the sizes of the cubes are obtained as a solution of a variational problem which is a discrete version of the notion of extremal length in $\mathbf{R}^3$.

0. INTRODUCTION

One exciting connection between two dimensional conformal geometry and packing is provided by the circle packing theorem, which asserts that if G is a finite planar graph, then there is a packing of Euclidean disks in the complex plane whose contact graph is G. This remarkable result was first proved by Koebe (cf. [13]) as a consequence of his theorem that every finite planar domain is conformally equivalent to a circle domain. Koebe's result was rediscovered and vastly generalized by Thurston (cf. [25] and [26, Chapter13]) as a corollary of Andreev's Theorem (cf. [2] and [3]). Thurston also conjectured that a sequence of maps, naturally associated to circle packings of a simply connected domain, converges to the Riemann map from this domain to the unit disk. This conjecture was proved by Rodin and Sullivan (cf. [15]), thus providing a second foundational connection between circle packing and conformal maps. In his thesis and later work, Schramm (see in particular [24, Theorem 6.1]) generalized the circle packing theorem allowing the tiles in the packing to be homothetic to C^1 closed topological disks. His result is based on an elaborated conformal uniformization theorem of Brandt and Harrington, and established another important connection between circle packing and two dimensional conformal geometry.

Following a suggestion by Thurston, Schramm (cf. [23]) studied the case in which the tiles in the packing are squares. An independent and similar study was carried out by Cannon, Floyd and Parry (cf. [9]), as part of their attempts to resolve Cannon's conjecture. Both results are based on *discrete extremal length* arguments, a notion first developed by Cannon (cf. [8]). This notion has its origin in the subject of two dimensional quasiconformal maps, where extremal length arguments are essential. There are a wealth of other results, relating combinatorics and packing that one should mention. Benjamini and Schramm (cf. [4]) studied the case where the tiled set is an infinite straight cylinder while Kenyon (cf. [12])

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allowed the tiles of the packing to be polygons. It is important to recall that the results in [12], as well as more contemporary work by the author of this paper (cf. [19] and [20]), do not use discrete extremal length methods. Rather, the usage of *discrete harmonic functions* as first employed by Dehn (cf. [11]), and later on by Brooks, Smith, Stone and Tutte (cf. [7]), is utilized. These results are also different from the ones obtained in [23], [9] and [4]. In these papers, a tile corresponds to a vertex in a given triangulation. In [11], [7], [12], [19] and [20], a tile corresponds to an edge of the triangulation. It is also worth noting that in [19] and [20], multi-connected, bounded, planar domains were studied (under the framework of boundary value problems on graphs) for the first time. This provides tiling of higher genus surfaces with conical singularities by rectangles.

The main goal of this paper is to provide the first connection between discrete extremal length in $\mathbf{R}^3$, a notion which we will recall in §1, and tiling by cubes (see Remark 3.31 for one possible generalization). In this paper, we study the case of a topological closed cube. It is interesting to note that quite recently Benjamini and Schramm (cf. [5]), as well as Benjamini and Curien (cf. [6]), have explored different and interesting applications of discrete extremal length in higher dimensions.

Before stating the main result of this paper we make

Definition 0.1. Let B be a closed triangulated topological ball, and let V, E and F denote the set of vertices, edges, and faces of the triangulation, respectively. Let $\partial B = B_1 \cup \bar{B}_1 \cup B_2 \cup \bar{B}_2 \cup B_3 \cup \bar{B}_3$ be a decomposition of ∂B in such a way that each B_j is a nonempty connected union of faces of the triangulation, $B_i \cap \bar{B}_j = \emptyset$ for $i = j$, and consists of a union of edges of the triangulation, if $i \neq j$. The collection $\mathcal{T} = \{V, E, F; B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3\}$ will be called a triangulation of a topological cube. We will denote by B_1 the *base face, by $\bar{B}_1$ the top face, by B_2 the front face, by $\bar{B}_2$ the back face, by B_3 the left face, and by $\bar{B}_3$ the right face.*

The main result of this paper is

Theorem 0.2. *Let $\mathcal{T} = \{V, E, F; B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3\}$ be a triangulation of a topological cube which has the triple intersection property. Then there exists a positive number h and a cube tiling $\mathcal{C} = \{C_v : v \in V\}$ of the rectangular parallelepiped $R = [0, \sqrt{h^{-1}}] \times [0, \sqrt{h^{-1}}] \times [0, h]$ such that*

$$C_v \cap C_u \neq \emptyset \text{ whenever } (v, u) \in E. \tag{0.3}$$

In addition, let $R_1, \bar{R}_1, R_2, \bar{R}_2, R_3,$ and $\bar{R}_3$ be the base, top, front, back, left, and right faces of R, respectively. Then it can also be arranged that for $i = 1, 2, 3$ we have

$$C_v \cap R_i(\bar{R}_i) \neq \emptyset \text{ whenever } v \in B_i(\bar{B}_i). \tag{0.4}$$

Under these conditions, the number h and the tiling $\mathcal{C}$ are uniquely determined.

Theorem 0.2 is a generalization to three dimensions of the main result (Theorem 1.3) as well as the techniques in [23], under an extra assumption. Schramm's proof (along with the proof given by Cannon, Floyd and Parry) fails to work in three dimension. The planarity of the triangulation is essential in their proofs. The triple intersection property (Definition 3.6), enables us to extract the ideas and techniques in [23] and carry out our proof, which then becomes straightforward. We have examples in which this property holds, and which can be characterized by saying that the triangulation has a *spine* (Definition 3.8).

The rest of this paper is organized as follows. In §1, we recall the notion of discrete extremal length in dimension three and prove the existence and uniqueness of an *extremal metric* (up to scaling). In §2, we prove that tiling by cubes induces an extremal metric, and in §3, we prove that an extremal metric induces a tiling by cubes. Finally, §4 is devoted to questions and suggestions for further research.

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1. Perspective and basic definitions

In the 1940's, Ahlfors and Beurling have refined existence methods (by Grötzch and Teichmüller) and extremal length was used as a conformally invariant measure of planar curve families (see for instance [1] for a useful account). Löwener (cf. [14]) showed how this method can be extended to three dimensions by defining a conformal capacity for rings in Euclidean 3-space by means of a Dirichlet integral. Väisälä (cf. [27, 28]) and Šabat (cf. [21, 22]) have used extremal length arguments to study quasiconformal mappings in 3-space, each of them has introduced a new kind of capacity for a ring in three dimensional Euclidean space. Shortly afterwards, Gehring (cf. [16, Theorem 1]) showed that slight modifications of their definitions are equivalent to the one given by Löwener. Extremal length arguments have continued to be useful tools in the theory of quasiconformal mappings of the plane and have found profound application in Teichmüller theory and the theory of hyperbolic manifolds and their deformations. We wish to restrict the background and preliminaries to a minimum. Hence, we will not describe many of the modern definitions and exciting applications of extremal length in the general setting of metric measure spaces. The interested reader is advised to consult for example Heinonen [17], and the references therein for an enjoyable and extensive account.

Our main result (Theorem 0.2) generalizes the main results of [23] and [9] that are based on Cannon's definition of two dimensional extremal length on a graph (cf. [8]). Cannon's definition was extended for arbitrary graphs in [23, Section 9]. The definition below is a special case of the one given in [23, Section 9], and is suitable to the applications of this paper. It is essentially a discrete version of the definition given by Väisälä. Let $G = (V, E)$ be a finite connected graph. A *path* in G is a sequence of vertices $\{v_0, v_1, \ldots, v_k\}$ such that any two successive vertices are connected by an edge. A nonnegative function $m : V \to [0, \infty)$ will be called a *metric* on G. Given a path $\alpha = \{v_0, v_1, \ldots, v_k\}$ in G and a metric m, we define the m-length of α as

$$l_m(\alpha) = \sum_{i=0}^{k} m(v_i). \tag{1.1}$$

Given $A_1, A_2 \subset V$, we define their m-distance to be

$$l_m(A_1, A_2) = \inf_{\alpha} l_m(\alpha), \tag{1.2}$$

where the infimum is taken over all paths α which join A_1 to A_2. The *volume* of the metric m is defined to be the cube of its 3-norm, i.e

$$\mathrm{vol}(m) = ||m||_3^3 = \sum_{v \in V} m(v)^3, \tag{1.3}$$

and the *normalized length* of (m, A_1, A_2) is defined as

$$\hat{l}_m = \frac{l_m^3}{\mathrm{vol}(m)}. \tag{1.4}$$

Finally, the *extremal three dimensional length* of (G, A_1, A_2) is defined by

$$\lambda(G; A_1, A_2) = \sup_m \hat{l}_m, \tag{1.5}$$

where the supremum is taken over the set of all metrics m with positive volume. An *extremal metric* for $(G; A_1, A_2)$ is one which realizes this supremum. Observe that for any positive constant c, $l_{cm} = c\, l_m$. Hence, $\hat{l}_m$ is a *conformal invariant* in this discrete setting.

An important theorem which asserts the existence and uniqueness (up to scaling) of an extremal metric was proved independently by Schramm and by Cannon, Floyd and Parry. We now recall the proof given by Cannon, Floyd and Parry (cf. [9, Theorem 2.2.1]), which applies (with negligible modifications) to our setting, in order to make this paper self-contained.

Theorem 1.6. *There is a unique extremal metric m_0 for (G, A_1, A_2) such that* $\mathrm{vol}(m_0) = 1$.

Proof. Let $\mathbf{N}$ denote the set of natural numbers, and let P denote a nonempty finite subset of $\mathbf{N}^n \setminus \{0\}$, where n is the cardinality of V. A path in G corresponds to an element in P. A metric m on G corresponds to a vector (which we will keep denoting by m) $m = (m_1, m_2, \ldots, m_n) \in \mathbf{R}^n \setminus \{0\}$, with $m_i \geq 0$ for $i = 1, \ldots n$. The length of a path with respect to the metric m is then given by the standard scalar product in $\mathbf{R}^n$. Since scaling does not change the extremal length $\lambda(G; A_1, A_2)$, we may restrict m to the subset of vectors of $\mathbf{S}^{n-1}$ in which each coordinate is nonnegative. Existence of an extremal metric now follows from the fact that the function which maps a metric $m \in \mathbf{S}^{n-1}$ to $l_m(A_1, A_2)$ is the minimum of a finite number of continuous functions; hence, it is continuous. It now easily follows that $\lambda(G; A_1, A_2)$ is attained and is positive. Uniqueness essentially follows since $(\mathbf{S}^{n-1}, \|.\|_3)$ is strictly convex. Given m_1, m_2 distinct nonnegative metrics in $\mathbf{S}^{n-1}$ such that

$$l_{m_1}(A_1, A_2) \geq l_{m_2}(A_1, A_2) \text{ and } t \in (0, 1), \text{ we let} \tag{1.7}$$

$$v = tm_1 + (1-t)m_2 \text{ (note that } 0 < ||v|| < 1). \tag{1.8}$$

Then for any path $p \in P$ we have

$$\begin{aligned} \frac{1}{||v||} < v, p > &= \frac{1}{||v||}(t < m_1, p > + (1-t) < m_2, p >) \\ &\geq \frac{1}{||v||}(t l_{m_1}(A_1, A_2) + (1-t) l_{m_2}(A_1, A_2)) \\ &\geq \frac{1}{||v||} l_{m_2}(A_1, A_2) > l_{m_2}(A_1, A_2), \end{aligned} \tag{1.9}$$

which clearly shows that the extremal metric (up to scaling) is unique. ∎

Remark 1.10. The geometry of discrete two dimensional extremal metrics was studied extensively by Parry and later on by Cannon, Floyd and Parry (see for example [8],[9]). We leave the study of the geometry of three dimensional extremal metrics for the future, since for the purposes of this paper only the assertion of Theorem 1.6 is needed.

Remark 1.11. An interesting recent reference by Wood (cf. [29]) explores some of the complications arising by the two inequivalent ways of carrying the notion of conformal modulus of a ring domain to a triangulated annulus. The first is by assigning a metric (as we do) to the vertices, and the second assigned a metric to the edges. It would be interesting to investigate this point in the three dimensional case.

2. Cube tilings give extremal metrics

In this section, we prove that a cube tiling of a rectangular parallelepiped yields in a natural way an extremal metric. Our proof is carried out by modifying the main idea of the proof of Lemma 4.1 in [23] to three dimensions. In order to ease the notation, and since we are working with fixed data, we let l_x denote $l_x(B_1, \bar{B}_1)$ in the lemma below.

Lemma 2.1. *Let $\mathcal{T} = \{V, E, F; B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3\}$ be as in Theorem 0.2, and suppose that h and $\mathcal{C}$ satisfy all the conditions there. Let $G = (V, E)$ be the 1-skeleton of $\mathcal{T}$, and let $s(v)$ denote the edge length of the square C_v. Then s is an extremal metric for $(G, B_1, \bar{B}_1)$.*

Proof. Let m be an arbitrary metric on G with positive volume. For every

$$(t, s) \in [0, \sqrt{h^{-1}}] \times [0, \sqrt{h^{-1}}], \tag{2.2}$$

let

$$\gamma_{t,s} = \{v \in V : \beta_{t,s} \cap C_v \neq \emptyset\}\,, \text{where } \beta_{t,s} = (t, s) \times \mathbf{R}. \tag{2.3}$$

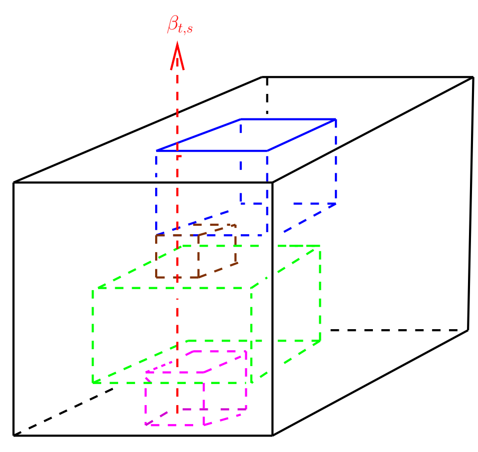

Figure 2.4. $\beta_{t,s}$ going through cubes in the tiling.

It is clear that $\gamma_{t,s}$ contains a simple path in G joining B_1 to $\bar{B}_1$. Hence, for every $(t, s) \in [0, \sqrt{h^{-1}}] \times [0, \sqrt{h^{-1}}]$ we have

$$l_m \leq \sum_{v \in \gamma_{t,s}} m(v). \tag{2.5}$$

We now integrate this inequality over $[0, \sqrt{h^{-1}}] \times [0, \sqrt{h^{-1}}]$ to obtain

$$(\sqrt{h^{-1}})^2 l_m \leq \int_0^{\sqrt{h^{-1}}} \int_0^{\sqrt{h^{-1}}} \sum_{v \in \gamma_{t,s}} m(v)\, dt\, ds. \tag{2.6}$$

Since $\mathcal{C}$ is a tiling and every $v \in V$ contributes $m(v)$ to the integral on the right hand side for an area measure of size $s(v)^2$, the integral can be rearranged to yield the following inequality

$$(\sqrt{h^{-1}})^2 l_m \leq \sum_{v \in V} m(v)s(v)^2. \tag{2.7}$$

We now let $s_1(v) = s(v)^2$ for every $v \in V$, and apply the Holder inequality with $p = 3$ and $q = 3/2$ to the above inequality to obtain

$$l_m \leq h \sum_{v \in V} m(v)s(v)^2 \leq h \,||m||_3||s_1||_{3/2} = h \,(\sum_{v \in V} m(v)^3)^{1/3}(\sum_{v \in V} s_1(v)^{3/2})^{2/3}. \tag{2.8}$$

Since $(\sum_{v \in V} s_1(v)^{3/2})^{2/3} = (\sum_{v \in V} s(v)^3)^{2/3}$ and $||s||_3^3 = \mathrm{vol}(R) = 1$, we finally obtain that

$$l_m \leq h \,||m||_3. \tag{2.9}$$

It is clear that $l_s = h$ and therefore that

$$\hat{l}_m = \frac{l_m^3}{||m||_3^3} \leq \frac{l_s^3}{||s||_3^3} = \hat{l}_s, \tag{2.10}$$

which implies that s is extremal.

□

3. Extremal metrics give cube tiling

The main result of this section is Theorem 3.18 which asserts that an extremal metric, under an extra condition imposed on a triangulation, induces cube tiling. We need several technical preparations before getting into the proof. Our proof is a generalization of the scheme in the two dimensional case given by Schramm (cf. [23]).

While a given metric on G is a discrete object in nature, Schramm (cf. [23, Section 5]) defined a continuous family of metrics which depends on a given curve.

Definition 3.1. Let α be any path in G and let m be any metric on G. For $t \geq 0$, we define a one parameter family of metrics on G by

$$m_{\alpha,t}(v) = \begin{cases} m(v) & \text{for } v \in V \setminus \alpha \\ m(v) + t & \text{for } v \in \alpha. \end{cases} \tag{3.2}$$

In the following, whenever the curve α is specified, we will use the notation m_t instead of $m_{\alpha,t}$.

Lemma 3.3. *For the family of metrics $m_t = m_{\alpha,t}$, $t \in [0, \infty)$ we have*

$$\frac{d}{dt}(||m_t||_3^3)\Big|_{t=0^+} = 3 \sum_{v \in \alpha} m(v)^2, \tag{3.4}$$

where α is any curve in G.

Proof. By definition

$$||m_t||_3^3 = \sum_{v \in V} m_t(v)^3 = \sum_{v \in V \setminus \alpha} m(v)^3 + \sum_{v \in \alpha}(m(v)^3 + 3m(v)^2 t + 3m(v)t^2 + t^3). \tag{3.5}$$

The assertion of the lemma follows by subtracting $||m||_3^3$ from the right hand-side of the equation above, dividing by $t > 0$, and taking the limit as $t \to 0^+$. □

Let $\mathcal{T}$ be a fixed triangulation of a closed topological cube Q, and we let $G = (\mathcal{T}^{(0)}, \mathcal{T}^{(1)})$ be the corresponding graph.

For the applications of this paper, we consider the following class of triangulations.

Definition 3.6. A triangulation of a closed topological cube Q will be said to have the triple intersection property, if the following property of the extremal metric m_0 of $(G, B_1, \bar{B}_1)$ holds. There exist a shortest m_0-path joining B_2 to $\bar{B}_2$ and a shortest m_0-path joining B_3 to $\bar{B}_3$ which meet all the shortest m_0 paths joining B_1 to $\bar{B}_1$.

In the figure below, the red curves correspond to all the shortest paths joining B_1 to $\bar{B}_1$, the yellow curve to shortest paths joining B_2 to $\bar{B}_2$, the green curve to a shortest paths joining B_3 to $\bar{B}_3$; all with respect to the m_0 metric.

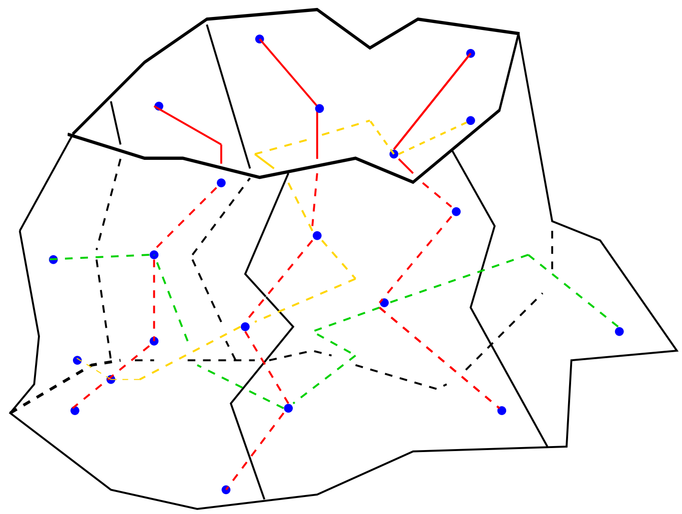

FIGURE 3.7. Shortest paths in a cube.

Once a triangulation is sufficiently tamed, in the sense described in the following definition, the triple intersection property will hold. It would be interesting to find if there are other classes of triangulations that have the triple intersection property (see for example Question 4.1). We keep the notation of Definition 0.1 and make

Definition 3.8. A triangulation of a closed topological cube Q will be said to have a spine if the following properties hold. There is one and only one path, called the *spine* of G, whose interior lies in B and its endpoints lie on $B_1, \bar{B}_1$, respectively. In addition, it is required that every path whose interior lies in B and joins B_2 to $\bar{B}_2$ or B_3 to $\bar{B}_3$, intersects the *spine* of G.

Suppose that $\mathcal{T}$ has the triple intersection property, and in addition that t is any nonnegative number which is smaller than the difference between a second shortest m_0 path and a shortest m_0 path, which join B_1 to $\bar{B}_1$. Consider any γ, δ shortest m_0-paths joining B_2 to $\bar{B}_2$ and B_3 to $\bar{B}_3$ as in Definition 3.6, respectively. Recall that the metrics $m_{\gamma,t}, m_{\delta,t}$ are obtained by adding t to $m_0(v)$ for each $v \in \gamma$, $v \in \delta$, respectively, and leaving other vertices with their m_0 values. Hence, for any t satisfying the condition above, by considering possible shortest paths for $m_{\gamma,t}, m_{\delta,t}$, it follows that

$$\min\{l_{m_{\gamma,t}}, l_{m_{\delta,t}}\} \geq l_{m_0} + t, \tag{3.10}$$

and therefore that

$$\min\{\frac{d}{dt}(l_{m_{\gamma,t}})|_{t=0^+}, \frac{d}{dt}(l_{m_{\delta,t}})|_{t=0^+}\} \geq 1. \tag{3.11}$$

Inequality (3.11) is essential for the applications of this paper. It will be used in the lemma below (Inequality (3.16)) which in turn is essential in the proof of the main theorem. We continue with

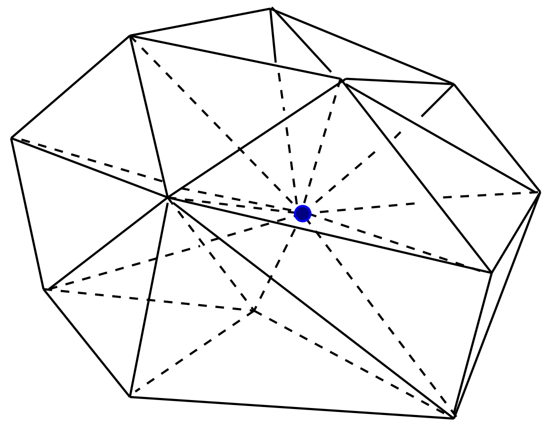

FIGURE 3.9. Part of a triangulation with a spine.

the following lemma which shows that shortest curves measured with respect to the extremal metric m_0 cannot be too short. This will be used in the proof of Theorem 3.18 to show that cubes arising from vertices that belong to boundary components of Q intersect appropriate (extended) boundary components naturally defined by R (for a more precise statement see the proof of Theorem 3.18).

Lemma 3.12. *With the notation and hypotheses of Theorem 0.2 and with m_0 being the extremal metric for $(G, B_1, \bar{B}_1)$ normalized so that* $\mathrm{vol}(m_0) = 1$*, we have*

$$\min\{l_{m_0}(\alpha), l_{m_0}(\beta)\} \geq \sqrt{h^{-1}}, \tag{3.13}$$

where $h = l_{m_0}$, α is any shortest m_0 path joining B_2 to $\bar{B}_2$, and β is any shortest m_0 path joining B_3 to $\bar{B}_3$.

Proof. Since m_0 is extremal, we have the following inequality for $l_{m_t} = l_{m_\alpha, t}$

$$0 \ \geq \ \frac{d}{dt}\left(\frac{(l_{m_t})^3}{||m_t||_3^3}\right)_{t=0^+} = \left(\frac{3l_{m_t}^2 \frac{d}{dt}(l_{m_t})||m_t||_3^3 - l_{m_t}^3 \frac{d}{dt}(||m_t||_3^3)}{||m_t||_3^6}\right)_{t=0^+}. \tag{3.14}$$

Hence, by applying Lemma 3.3 and the normalization $\mathrm{vol}(m_0) = 1$, we must have

$$0 \geq \left(l_{m_t}^2 \frac{d}{dt}(l_{m_t}) - l_{m_t}^3 \sum_{v \in \alpha} m_0(v)^2\right)_{t=0^+}, \tag{3.15}$$

which implies, if α satisfies Inequality (3.11), that

$$\sum_{v \in \alpha} m_0(v)^2 \geq l_{m_0}^{-1} = h^{-1}. \tag{3.16}$$

Since

$$l_{m_0}(\alpha)^2 = (\sum_{v \in \alpha} m_0(v))^2 \geq \sum_{v \in \alpha} m_0(v)^2, \tag{3.17}$$

and all m_0 shortest paths joining B_2 to $\bar{B}_2$ have the same length, the assertion of the lemma follows (an identical argument holds for β).

□

We do not know if inequality (3.11) is necessary for the assertions of Theorem 0.2 to hold. It is definitely necessary in our proof of the lemma above, as well as in the analogous part of Schramm's proof of his main theorem. The triple intersection property guarantees that this inequality holds.

We now turn into the construction of cube tiling from a normalized extremal metric.

Theorem 3.18. *Let $\mathcal{T} = \{V, E, F; B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3\}$ be a triangulation of a topological cube which has the triple intersection property, and let $G = (V, E)$ be the 1-skeleton of $\mathcal{T}$. Let m be the extremal metric for $(G, B_1, \bar{B}_1)$ normalized so that* $\mathrm{vol}(m) = 1$. *Set*

$$h = l_m, \text{ and let } R = [0, h] \times [0, \sqrt{h^{-1}}] \times [0, \sqrt{h^{-1}}]. \tag{3.19}$$

For each $v \in V$ let

$$C_v = [x(v) - m(v), x(v)] \times [y(v) - m(v), y(v)] \times [z(v) - m(v), z(v)], \tag{3.20}$$

where $x(v)$ (respectively $y(v), z(v)$) is the least m-length of paths from $\bar{B}_2$ (respectively, B_3, B_1) to v. Then $\mathcal{C} = \{C_v : v \in V\}$ is a cube tiling of the rectangular parallelepiped R which satisfies the contact and boundary constraints (0.3) *and* (0.4)*.*

We may now turn to the proof of our main theorem.

Proof of Theorem 0.2. The existence of a cube tiling follows from the existence part in Theorem 1.6 and from Theorem 3.18. Uniqueness follows from Lemma 2.1 and the uniqueness part in Theorem 1.6.

$\boxed{\mathbf{0.2}}$

We now turn to the

Proof of Theorem 3.18. We start by showing that the combinatorics is preserved in the sense of constraint (0.3). Let $(u, v) \in \mathcal{T}^{(1)}$ be given. We claim that

$$x(v) - m(v) \le x(u) \text{ and } x(u) - m(u) \le x(v). \tag{3.21}$$

Suppose that $x(v) - m(v) > x(u)$, and let α_u be a shortest m path joining u to $\bar{B}_2$. Then the path $[v, u] \cup \alpha_u$ which joins v to $\bar{B}_2$ has m-length which is equal to $m(v) + x(u) < x(v)$. This is absurd. Hence, by applying a symmetric argument to prove the second inequality, we have that

$$[x(v) - m(v), x(v)] \cap [x(u) - m(u), x(u)] \ne \emptyset. \tag{3.22}$$

The argument above goes through for the coordinates $y(v)$ and $z(v)$ (up to replacing $\bar{B}_2$ with B_3 and B_1, respectively). Thus, as claimed

$$Z_v \cap Z_u \ne \emptyset. \tag{3.23}$$

We now define several rectangular parallelepipeds in $\mathbf{R}^3$ where some are degenerate, and the remaining are infinite and all of which are naturally associated with R.

Let $R_1 = \{(x, y, z) : \min\{x, y\} \ge 0, z = 0\}$, $\hat{R}_1 = \{(x, y, z) : \min\{x, y\} \ge 0, z \ge h\}$, $R_2 = \{(x, y, z) : \min\{y, z\} \ge 0, x \ge \sqrt{h^{-1}}\}$, $\hat{R}_2 = \{(0, y, z) : \min\{y, z\} \ge 0\}$, $R_3 = \{(x, 0, z) : \min\{x, z\} \ge 0\}$, and $\hat{R}_3 = \{(x, y, z) : \min\{x, z\} \ge 0, y \ge \sqrt{h^{-1}}\}$.

Since the volume of R is equal to one, which by assumption is also equal to $\mathrm{vol}(m)$, in order to prove that $\mathcal{C}$ tiles R, it suffices to prove the following

$$R \subset \bigcup_{v \in V} C_v, \tag{3.24}$$

for then it follows that there are no overlaps of positive volume among the cubes and no cube extends beyond R. To this end we first prove

Lemma 3.25. *With the notation above we have that ∂R is freely homotopic to a constant in*

$$\bigcup_{v \in V} C_v \cup \left(\mathbf{R}^3 \setminus \mathrm{int}(R)\right). \tag{3.26}$$

Proof. We begin by constructing a map $f : \mathcal{T} \to \bigcup_{v\in V} C_v$. For each $v \in V$, choose $f(v) \in C_v$ such that $f(v) \in R_i(\hat{R}_i)$ for $i = 1, 2, 3$, and whenever $v \in B_i(\bar{B}_i)$. We observe that this may be done in a consistent way; that is, whenever v is in the intersection of two or three faces among $\{B_i, \bar{B}_i\}$, i=1, 2, 3, then the corresponding intersection among the $\{R_i, \hat{R}_i\}$ is nonempty.

We endow $\mathcal{T}$ with a piecewise linear structure by declaring that each 3-dimensional face (u, v, w, s) of $\mathcal{T}^{(3)}$ is linearly parametrized by a regular tetrahedron, all of its edges have length 1, and that these parametrizations are compatible along faces. For each 1-dimensional face $(u, v) \in \mathcal{T}^{(1)}$, let $m_{(u,v)}$ denote the midpoint of this edge; for each 2-dimensional face $(u, v, w) \in \mathcal{T}^{(2)}$, let $c_{(u,v,w)}$ denote the barycentric center of this face, and for each three dimensional tetrahedron $(u, v, w, s) \in \mathcal{T}^{(3)}$, let $p_{(u,v,w,s)}$ denote its barycentric center.

Choose $f(m_{(u,v)})$ to be some point in $C_v \cap C_u$, choose $f(c_{(u,v,w)})$ to be some point in $C_v \cap C_u \cap C_w$, and choose $f(p_{(u,v,w,s)})$ to be some point in $C_u \cap C_v \cap C_w \cap C_s$. The first choice is possible by applying (3.23), the second and the third choices are possible due to the fact that if three (four) cubes whose edges are parallel to the coordinate axes have the property that the intersection of any two (three) of these cubes is nonempty, then the intersection of the three (four) cubes is nonempty. We also require that $f(m_{(u,v)}) \in R_i(\hat{R}_i)$ if $u, v \in B_i(\bar{B}_i)$, and that $f(c_{(u,v,w)}) \in R_i(\hat{R}_i)$ if $u, v, w \in B_i(\bar{B}_i)$.

Let $\mathcal{T}^*$ be the first barycentric subdivision of $\mathcal{T}$, and extend f by requiring it to be affine on each face of $\mathcal{T}^*$. It is clear (by construction) that the extension is well defined. Also, since for each face $(s, m_{s,u}, c_{s,u,v}, p_{s,u,v,w})$ of $\mathcal{T}^*$ the four points $f(s), f(m_{s,u}), f(c_{s,u,v})$, and $f(p_{s,u,v,w}))$ lie in C_s which is convex. Hence

$$f(\mathcal{T}) \subset \bigcup_{v\in V} C_v. \tag{3.27}$$

Let v be any vertex in $\mathcal{T}^{(0)}$, then it is clear that $\partial\mathcal{T}$ is freely homotopic in $\mathcal{T}$ to v. By construction, $f(\partial\mathcal{T})$ is freely homotopic to ∂R in $\mathbf{R}^3 \setminus \mathrm{int}(R)$. Note that the last part is justified (in part) due to the assertions of Lemma 3.12. It is here where we are using in an essential way a lower bound for the shortest m-curves joining B_2 to $\bar{B}_2$ and B_3 to $\bar{B}_3$. The assertion of the lemma follows immediately by defining the constant to be $f(v)$, and composing the two homotopies above.

$\boxed{\mathbf{3.25}}$

We now finish the proof of the theorem by establishing (3.24). We argue by contradiction. First suppose that there exists a point

$$x \in \mathrm{int}(R) \text{ such that } x \notin \bigcup_{v\in V} C_v. \tag{3.28}$$

Thus, we have the inclusion

$$\bigcup_{v\in V} C_v \cup \big(\mathbf{R}^3 \setminus \mathrm{int}(R)\big) \hookrightarrow \mathbf{R}^3 \setminus \{x\}. \tag{3.29}$$

Hence, by the assertion of the previous lemma, $\partial R \simeq \mathbf{S}^2$ is homotopic to a constant in $\mathbf{R}^3 \setminus \{x\}$. This is absurd. To end, one treats the case $x \in \partial R$ and $x \notin \bigcup_{v\in V} C_v$, by arguing that since $\bigcup_{v\in V} C_v$ is a closed set, there exists a point $y \in \mathrm{int}(R)$ which is close to x and is not in $\bigcup_{v\in V} C_v$.

$\boxed{\mathbf{3.18}}$

Remark 3.30. Since at most eight cubes may be tiled in $\mathbf{R}^3$ with a nonempty intersection, it is feasible that some cubes in the tiling provided by Theorem 3.18 will degenerate to points.

Remark 3.31. There are straightforward modifications of our definitions and proofs that allow generalizations of the results to tiling with rectangular parallelepipeds of specified aspect ratios.

In the two dimensional case, one such generalization (tiling by rectangles instead of squares) was observed by Schramm (cf. [23, Section 8]).

Let $\omega : V \to (0,\infty)$ be some assignment of weights to the vertices, and for every metric $m : V \to [0,\infty)$ on $\mathcal{T}$ define the ωm length of a path $\alpha = (v_0, v_1, \ldots, v_k)$ as

$$l_{\omega m}(\alpha) = \sum_{i=0}^{k} \omega(v) m(v_i),$$

and the ω-volume by

$$\|m\|_\omega^3 = \sum_{v \in V} \omega(v) m(v)^3.$$

Define the ω-extremal length of $\mathcal{T}$ to be

$$\lambda(G, B_1, \bar{B}_1) = \sup_m \frac{l_m^3}{\|m\|_\omega^3},$$

where the supremum is taken over all metrics of positive area. Then, as before $\lambda(G, B_1, \bar{B}_1)$ is a (discrete) conformal invariant, and a ω-extremal metric always exists, and is unique up to a positive scaling factor. With this setting, the following holds (we omit the straightforward details of the proof as well as other possible generalizations).

Theorem 3.32. *Let $\mathcal{T} = \{V, E, F; B_1, \bar{B}_1, B_2, \bar{B}_2, B_3, \bar{B}_3\}$ be a triangulation of a topological cube which has the triple intersection property, and let $G = (V, E)$ be the* 1*-skeleton of $\mathcal{T}$. Let $\omega : V \to (0,\infty)$ be some assignment of weights to the vertices. Let m be the ω-extremal metric for $(G, B_1, \bar{B}_1)$ that satisfies $\|m\|_\omega^3 = 1$. Set*

$$h = l_m, \text{ and let } R = [0,h] \times [0, \sqrt{h^{-1}}] \times [0, \sqrt{h^{-1}}]. \tag{3.33}$$

For each $v \in V$ let

$$C_v = [x(v) - \sqrt{\omega(v)}m(v), x(v)] \times [y(v) - \sqrt{\omega(v)}m(v), y(v)] \times [z(v) - m(v), z(v)], \tag{3.34}$$

where $z(v)$ (respectively, $x(v), y(v)$) is the least m-length of a path from v to B_1 (least ωm-length to $\bar{B}_2, B_3$, respectively). Then $\mathcal{C} = \{C_v : v \in V\}$ is a tiling of the rectangular parallelepiped R which satisfies the contact and boundary constraints (0.3) *and* (0.4).

4. Further questions and research directions

We end this paper by suggesting several future research directions and questions that are motivated in part by the extensive study done in the two dimensional case (see for example [4, 11, 8, 9, 10, 12]).

Definition 3.6 specifies a class of triangulations of a closed topological cube for which the assertions of Theorem 0.2 holds.

Question 4.1. Are there larger classes of triangulations which induce a tiling by cubes?

Experience shows that the method of extremal length is very useful when two boundary components are chosen (these are the top base and the bottom base in our work). The passage for 3-manifolds without boundary invites further investigations.

Question 4.2. What is the analogue of Theorem 0.2 for a ring space domain, and even more generally for a genus g handlebody?

The works in [23] and in [9] contain various algorithms to compute two dimensional extremal length for a triangulation of a quadrilateral. All of these use the planarity in an essential way. The following seems to be quite natural to pose.

Question 4.3. Is there an efficient algorithm to compute extremal length for a given $\mathcal{T}$?

There is a combinatorial notion of *a boundary value* data which may naturally be associated with a cube tiling, that is, the induced square tiling of the faces. We propose

Question 4.4. Given a pattern of square tiling of some (perhaps all of) the faces of R, does there exist a cube tiling of R that induces this pattern?

Wood ([29, 30]) studied how two dimensional discrete extremal length and the associated modulus changes under various effects of combinatorial operations on a triangulated planar annulus, and related questions on triangulated Riemann surfaces. Without getting into technical definitions, we pose the following.

Question 4.5. What are the effects of (for example) refinement of a triangulation on the discrete three dimensional extremal length? (We do not have a good understanding of this even in the case discussed in this paper.)

We close this list of questions by one which is motivated by the classical continuous theory of extremal length. Due to the work of various authors (see the beginning of §1), there are intimate relations between extremal length and harmonic functions. The work in [18, 19, 20] shows that the classical theory does not transform word by word to the discrete setting, tiling by cubes which is induced by harmonic maps is possible, yet more complicated to construct.

Question 4.6. Assume that $\mathcal{T}$ is given (for a topological cube or a handlebody), does there exist a tiling by cubes (or by rectangular parallelepipeds) which is generated by the discrete harmonic function defined on V and which satisfies suitable combinatorial boundary conditions (such as Dirichlet or Dirichlet-Neumann)?

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