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ON THE VOLUMES OF COMPLEX HYPERBOLIC MANIFOLDS

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1. Introduction. Let M be a complete locally symmetric Riemannian manifold of negative curvature. The main goal of this paper is to give estimates on the smallest volumes of such M's. We will do that in the complex case, but quaternionic or octonion analogues of Theorem 5.1 are true.

Fix $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{Ca}\}$ the real, complex, or quaternionic field, or the Cayley octonion algebra, and $n \geqslant 2$ an integer with $n = 2$ if $\mathbf{K} = \mathbf{Ca}$. By a theorem of E. Cartan, the universal cover of M is isometric to $\mathbf{H}_{\mathbf{K}}^n$, the hyperbolic space over $\mathbf{K}$ of dimension n. By a theorem of H. C. Wang (see [Wan, Theorem 8.1]), if $(\mathbf{K}, n) \neq (\mathbf{R}, 2), (\mathbf{R}, 3)$, and of Jorgensen-Thurston (see [Gro2]) if $(\mathbf{K}, n) = (\mathbf{R}, 3)$, there does exist a manifold covered by $\mathbf{H}_{\mathbf{K}}^n$ of smallest volume. Moreover, the minimum is obtained by only finitely many manifolds (up to isometry).

The *closed* real hyperbolic 3-manifold of smallest known volume is the J. Weeks and S. Matveev-A. Fomenko manifold, having volume ≈ 0.94272. The best-known lower bound is ≈ 0.00115, due to F. Gehring-G. Martin [GM]. Of related interest is the work of M. Culler-P. Shalen (with P. Wagreich, J. Anderson-R. Canary, S. Hersonsky) proving, for example, that every real hyperbolic 3-manifold of smallest volume has first Betti number less than or equal to 2 [CHS]. Note that the smallest volume of a noncompact real hyperbolic 3-manifold and 3-orbifold are, respectively, σ (C. Adams [Ada]) and $\sigma/24$ (R. Meyerhoff [Mey1]), where σ is the volume of the regular ideal real hyperbolic tetrahedra, $\sigma \approx 1.0149414$.

If the real dimension of M is even and if M has finite volume, since $\mathbf{H}_{\mathbf{K}}^n$ is homogeneous, the Gauss-Bonnet formula (see [Spi, vol. 4, page 443]; as extended by Harder-Gromov [Gro3, page 84] in the noncompact case; see also [Mum1]; or Hirzebruch proportionality theorem [Hir3, Theorem 22.2.1]) tells us that there is a constant $\kappa_{\mathbf{K},n}$ such that

$$\operatorname{vol}(M) = \kappa_{\mathbf{K},n}\, \chi_{\mathrm{top}}(M)$$

with $\chi_{\mathrm{top}}(M)$ the Euler characteristic of M. The exact value of the constant has been explicitly computed, for instance in [Hir1], giving in the complex case, when the holomorphic sectional curvature is normalized to be -1 (hence the sectional

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curvature is between -1 and $-1/4$)

$$\kappa_{\mathbf{C},n} = \kappa_n = \frac{(-\pi)^n 2^{2n}}{(n+1)!} .$$

In particular, the Euler characteristic of a complex hyperbolic n-manifold is positive if and only if n is even.

If $(\mathbf{K}, n) = (\mathbf{R}, 2), (\mathbf{R}, 4)$, with $\kappa_{\mathbf{R},2} = -2\pi$ and $\kappa_{\mathbf{R},4} = 4\pi^2/3$, the smallest $|\chi_{\text{top}}(M)|$ is 1 (J. Ratcliffe-S. Tschantz [RaTs] for the last case). In general, finding the smallest volume of M is as difficult as finding the smallest Euler characteristic (in absolute value) of M. For instance, there apparently does not exist (though see [Hol] for complex hyperbolic surfaces and [Lan]) a general method for computing the Euler characteristic of arithmetic manifolds. These were first constructed by [BoHC, Bor]. They are the only known examples of such M with finite volume, if $\mathbf{K} \neq \mathbf{R}$, and if $\mathbf{K} = \mathbf{C}$ and $n \geqslant 3$, and the only possible examples if $\mathbf{K} \neq \mathbf{R}, \mathbf{C}$ (see [GS]).

Our first main result, also known by P. Pansu (private communication), is the following theorem.

THEOREM 2.1. *The smallest volume (resp., Euler characteristic) of a closed, complex hyperbolic surface is* $8\pi^2$ *(resp., 3).*

The number of manifolds having this volume is unknown. We will prove that any closed, complex hyperbolic surface has Euler characteristic a multiple of 3. It is unknown which multiple can be obtained. By F. Hirzebruch's example Y_1 [Hir1, page 134], all multiples of 15 are obtained. Indeed, Y_1 is a closed complex hyperbolic surface, having by a result of M.-N. Ishida [Ish, Section 6, Example 5] a group G of order 125 acting freely on it with quotient Y_1/G having Euler characteristic 15 and nonzero first cohomology group. Hence the fundamental group of Y_1/G maps onto a finite group of order n for all n. Taking the associated finite covers gives the answer. This argument does not work for the Mumford example [Mum2] (which is the only known example with minimal Euler characteristic), since its first cohomology group is 0.

The analogue of Theorem 2.1 in the nonclosed case is unknown. In particular, we do not know yet whether there exists a finite volume (noncompact) complex hyperbolic surface of Euler characteristic 1.

Let us denote by $\mathbf{Heis}_{2n-1}$ the Heisenberg group of dimension $2n-1$ endowed with any left invariant Riemannian metric. Let Γ be any discrete, cocompact, torsion-free subgroup of the full group of isometries of $\mathbf{Heis}_{2n-1}$. It is well known (see, for example, [Gro1], [BK, Theorem 1]) that Γ contains a cocompact lattice of $\mathbf{Heis}_{2n-1}$ with index bounded above by a universal constant I_n. Note that the analogue in the Euclidean case is the Bieberbach theorem. For $n = 2$, we obtain in Proposition 5.8 that $I_2 = 6$.

Our second main result is the following theorem.

THEOREM 5.1. *Let M be a complex hyperbolic n-manifold of finite volume with k ends. Then*

$$\mathrm{vol}(M) \geqslant \frac{k}{nI_n}.$$

This is a complex-dimensional analogue of [Her1, Theorem 1], which gives a lower bound for the volume of a real hyperbolic n-manifold with cusps. There is an analogous theorem for orbifolds, only the constant I_n is changing.

J. Parker informed us that it is possible to improve our lower bound k/nI_n by a factor of 2^n and even by a factor of 12 for $n = 2$ and $k = 1$; see [Par4].

Note that for $n = 2$, this lower bound is weaker than the one given by the Gauss-Bonnet formula, which predicts that the volume has to be at least $8\pi^2/3$ (see Section 3). The previous result gives only $1/12$ when M has one end. But we have the immediate following fact.

COROLLARY. *The number of ends of a finite-volume complex hyperbolic n-manifold is at most*

$$n\kappa_n I_n \, \chi_{\mathrm{top}}(M) \; .$$

Since $I_n \leqslant 2(6\pi)^{(2n-1)(n-1)}$ (see [BK], page 10), one may get an estimate of the above constant.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 2.1. The main tool is the proportionality principle of Hirzerbruch mentioned above. In Section 3, we briefly review well-known notions related to complex hyperbolic space. In Section 4, we introduce the Ford isometric spheres and Cygan metric developed by W. Goldman [Gol] and J. Parker [Par1, Par2]. In Section 5, we treat the noncompact case and prove Theorem 5.1, following the same strategy as in [Her1]. We use the generalization of the classical Shimizu inequality (for PSL(2, **C**)) by [Kam1], [Par3] to the complex hyperbolic case to find the maximal *horoball* quotient that we can embed in the manifold. We then approximate the volume of such a quotient. In Section 5, we prove a generalization of the classical Shimizu inequality for complex hyperbolic space, simplifiying computations of [Par3].

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2. The smallest volume of a closed complex hyperbolic surface. This section is devoted to the proof of our first main result.

THEOREM 2.1. *The smallest volume of a closed complex hyperbolic surface M is $8\pi^2$.*

Proof. It follows from the introduction that

$$\mathrm{vol}(M) = \frac{8\pi^2}{3}\,\chi_{\mathrm{top}}(M).$$

Theorem 2.1 then follows from the next result. □

PROPOSITION 2.2. *The smallest Euler characteristic of a closed complex hyperbolic surface M is* 3.

Proof. Note that the group of biholomorphisms of the unit ball of $\mathbf{C}^2$ is precisely the group of orientation preserving isometries for the Bergmann metric. Hence the compact, complex surfaces covered by the unit ball of $\mathbf{C}^2$ are precisely the closed, compact, oriented, complex hyperbolic surfaces.

First assume M is oriented, with fundamental class $[M]$. Let c_1, c_2 be the Chern classes of the compact complex surface M, with $\langle c_2, [M]\rangle = \chi_{\mathrm{top}}(M)$. Since M is covered by the unit ball, F. Hirzebruch [Hir2] proved that

$$c_1^2 = 3c_2.$$

By E. Noether's formula for closed complex surfaces (see, for instance, [BPV, page 20]), we have

$$c_1^2 + c_2 \equiv 0 \pmod{12}.$$

Hence, $\chi_{\mathrm{top}}(M)$ must be a multiple of 3. In the case M is not oriented, we pass to the oriented double cover M_1 of M. By the above arguments, $\chi_{\mathrm{top}}(M_1)$ must be a multiple of 3, and even; therefore $\chi_{\mathrm{top}}(M) \geqslant 3$.

To finish the proof of the proposition, we note that D. Mumford (see [Mum2]) has constructed a closed complex surface, covered by the ball, such that $\chi_{\mathrm{top}}(M) = 3$. □

3. The complex hyperbolic *n*-space. In this section, we recall some basic facts regarding complex hyperbolic space. The reader is referred to [CG], [Eps], and [Gol] for the details and further important results on the subject. We mainly follow the presentation of isometries of $\mathbf{H}^n_{\mathbf{C}}$ given in [CG].

Let

$$q = -(z_0\bar{z}_1 + z_1\bar{z}_0) + z \cdot \bar{z} \tag{1}$$

be our chosen hermitian form of signature $(n, 1)$, defined on $\mathbf{C}^{n+1} = \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$, with coordinates (z_0, z_1, z) (where $z \cdot \bar{z}$ is the standard hermitian form on $\mathbf{C}^{n-1}$).

Define as usual $E^* = \bar{E}^t$ for any $m \times n$ matrix E. We denote by

$$X = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & A \end{pmatrix} \tag{2}$$

a generic matrix in the unitary group $U(q)$, with the decomposition into blocks induced by the above splitting of $\mathbf{C}^{n+1}$. Since X preserves the form q, one has that $X^{-1} = Q^{-1}X^*Q$, where Q is the matrix representing q in the canonical basis. Therefore

$$X^{-1} = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & A^* \end{pmatrix}. \tag{3}$$

As a model for the complex hyperbolic space of dimension n, we take the Siegel upper half-space model. That is, we have

$$\mathbf{H}^n_{\mathbf{C}} = \{(w_1, w) \in \mathbf{C} \times \mathbf{C}^{n-1} : 2 \operatorname{Re} w_1 - |w|^2 > 0\},$$

endowed with the Riemannian metric defined by

$$ds^2_{\mathbf{C}} = \frac{4}{(2 \operatorname{Re} w_1 - |w|^2)^2}((dw_1 - dw \cdot \bar{w})(\overline{dw_1} - w \cdot \overline{dw}) + (2 \operatorname{Re} w_1 - |w|^2)\, dw \cdot \overline{dw}), \tag{4}$$

which has constant holomorphic sectional curvature -1. Note that for $t > 0$, the map $\phi_t : (w_1, w) \mapsto (t^2 w_1, tw)$ is an isometry of the Siegel domain, called a *dilatation.*

The *projective model* of the complex hyperbolic n-space is obtained by mapping $\mathbf{H}^n_{\mathbf{C}}$ into the complex projective space $\mathbf{P}^n(\mathrm{C})$ via the map $(w_1, w) \to [1, w_1, w]$ (the standard homogeneous coordinates on $\mathbf{P}^n(\mathrm{C})$). The image of $\mathbf{H}^n_{\mathbf{C}}$ under the above map corresponds precisely to the open cone defined by $q < 0$. Hence, $\mathrm{PU}(q)$ acts naturally on $\mathbf{H}^n_{\mathbf{C}}$.

It is well known that $\mathrm{PU}(q)$ is the group of orientation preserving isometries of $\mathbf{H}^n_{\mathbf{C}}$, and that $\mathrm{PU}(q)$ acts transitively on the unit tangent bundle of $\mathbf{H}^n_{\mathbf{C}}$. Let us remark that $\mathrm{PU}(q)$ is also the group of biholomorphic automorphisms of the Siegel domain. Although we will barely need it, also note that the Siegel upper half-space is biholomorphic to the unit ball of $\mathbf{C}^n$; hence there is also a ball model for the complex hyperbolic n-space.

The *space at infinity* is defined by

$$\partial \mathbf{H}^n_{\mathbf{C}} = \{(w_1, w) \in \mathbf{C} \times \mathbf{C}^{n-1} : 2 \operatorname{Re} w_1 - |w|^2 = 0\} \cup \{\infty\},$$

where ∞ corresponds to $[0, 1, 0]$ in $\mathbf{P}^n(\mathrm{C})$). Hence $\mathrm{PU}(q)$ acts transitively on $\partial \mathbf{H}^n_{\mathbf{C}}$ (via the identification with geodesic rays starting from a basepoint and the space at infinity).

In $\mathbf{H}^n_{\mathbf{C}}$ the subspaces defined by

$$H_t = \{(w_1, w) \in \mathbf{C} \times \mathbf{C}^{n-1} : 2 \operatorname{Re} w_1 - |w|^2 \geqslant t\}$$

are called *horoballs* and

$$\partial H_t = \{(w_1, w) \in \mathbf{C} \times \mathbf{C}^{n-1} : 2 \operatorname{Re} w_1 - |w|^2 = t\}$$

are called *horospheres.* The subgroup in $\mathrm{PU}(q)$ which preserves each horosphere is precisely the subgroup of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ A\zeta & 0 & A \end{pmatrix}$$

with $v \in \mathbf{R}$, $\zeta \in \mathbf{C}^{n-1}$ and $A \in \mathbf{U}(n-1)$. It is easy to see that the subgroup of $U(q)$ fixing the point ∞ is projectively generated by the above group and the dilatations ϕ_t.

Consider the Heisenberg group $\mathbf{Heis}_{2n-1}$ which is the set $\mathbf{C}^{n-1} \times \mathbf{R}$ (with coordinates (ζ, v)) endowed with the multiplication law (with the usual conventions of Koranyi, Reimann, Goldman, and Parker)

$$(5) \qquad (\zeta, v) \cdot (\zeta', v') = (\zeta + \zeta', v + v' + 2 \operatorname{Im} \zeta \cdot \bar{\zeta}').$$

There exists a central extension

$$(6) \qquad 0 \to \mathbf{R} \to \mathbf{Heis}_{2n-1} \to \mathbf{C}^{n-1} \to 0$$

making $\mathbf{Heis}_{2n-1}$ a simply connected nilpotent Lie group of order 2 and dimension $2n-1$. The center of $\mathbf{Heis}_{2n-1}$ is obviously $\{0\} \times \mathbf{R}$.

For $g, h \in \mathbf{Heis}_{2n-1}$ we denote by $[g, h]$ the commutator of g and h in $\mathbf{Heis}_{2n-1}$.

LEMMA 3.1 *For any $g, h \in \mathbf{Heis}_{2n-1}$, we have*

$$[g, h] = (0, -4\omega(g, h)),$$

where ω is the standard symplectic form on $\mathbf{C}^{n-1}$. In particular, $[g, h]$ is a vertical translation.

Proof. The proof follows easily from the definition of the multiplication in the Heisenberg group, given in equation (5). □

We will use the identification of $\mathbf{Heis}_{2n-1}$ with a subgroup of $\mathrm{PU}(q)$ fixing the point at infinity, defined by

$$(\zeta, v) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}$$

where I is the identity matrix in any dimension. We then have that $\mathbf{Heis}_{2n-1}$ is acting on $\mathbf{H}^n_{\mathbf{C}}$ by the formula

$$(\zeta, v)(w_1, w) = \left(w_1 + w \cdot \overline{\zeta} + \frac{1}{2}|\zeta|^2 - \frac{i}{2}v, w + \zeta\right).$$

The action is simply transitive on each horosphere. Elements of the form $(0, v)$ are called *vertical translations.*

We will need the following discreteness criterion in Section 5. It is a complex analogue of the classical Shimizu inequality for discrete subgroups of $\mathrm{PSL}(2, \mathbf{C})$, as generalized by [Her2] and [Wat] to discrete subgroups of the Möbius group in any dimension. This generalization played a fundamental role in [Her1] to give volume estimates mentioned before. In [Wat] it was used to give universal constraints on the radii of isometric spheres in discrete groups on Möbius transformations of all dimensions.

THEOREM 3.2 (S. Kamiya [Kam1, Theorem 3.2], J. Parker [Par3, Proposition 5.2]). *Let G be a discrete nonelementary subgroup of $U(q)$. Suppose that*

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{i}{2}v & 1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

belongs to G with $v \neq 0$. Then for every X with $X \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in G, $|b| \geqslant 2/|v|$.

4. Cygan metric and isometric spheres. In [Gol] Goldman extends the definition of isometric spheres of Möbius transformations acting on the upper half-space to the *Ford isometric spheres* of complex hyperbolic transformations of the Siegel domain. These spheres and their associated geometric properties have been extensively used in [Gol], [Par1], [Par2], [Par3]. Since we are using the hermitian form q, defined in equation (1), which is different from the Goldman-

Parker one, we will briefly describe the suitably modified isometric sphere. Let $\Omega = (0, 1, 0)$ be a point of $\mathbf{C}^{n+1}$ corresponding to the point at infinity.

Definition 4.1. Let $X \in U(q)$. Suppose that X does not fix Ω. Then the isometric sphere of X is the hypersurface

$$I_X = \{z \in \mathbf{H}^n_{\mathbf{C}}: |q(Z, \Omega)| = |q(Z, X^{-1}\Omega)|\},$$

for any $Z \in \mathbf{C}^{n+1}$ which maps onto z projectively.

The above equality does not depend on the choice of Z. As in the real case, X maps I_X to $I_{X^{-1}}$ and X maps the component of $\overline{\mathbf{H}^n_{\mathbf{C}}} - I_X$ containing ∞ to the component of $\overline{\mathbf{H}^n_{\mathbf{C}}} - I_{X^{-1}}$ not containing ∞.

We will also need the following *horospherical coordinates on* $\overline{\mathbf{H}^n_{\mathbf{C}}} - \{\infty\}$, introduced in [GoPa]. Given a point (w_1, w), they are defined, using the unique (ζ, v) in the Heisenberg group such that the image of $(t/2, 0)$ by the Heisenberg element is (w_1, w), as follows.

Definition 4.2. The horospherical coordinates of a point $(w_1, w) \in \mathbf{H}^n_{\mathbf{C}}$ are $(\zeta, v, t) \in \mathbf{C}^{n-1} \times \mathbf{R} \times \mathbf{R}$ given by

$$\zeta = w, \ v = -2 \operatorname{Im} w_1 \qquad \text{and} \qquad t = 2 \operatorname{Re} w_1 - |w|^2$$

so that

$$w_1 = \frac{1}{2}(|w|^2 + t) - \frac{i}{2} v \qquad \text{and} \qquad w = \zeta.$$

In [Par1], an extension of the *Cygan metric on* $\mathbf{Heis}_{2n-1}$ was carried to $\mathbf{H}^n_{\mathbf{C}}$ endowed with the horospherical coordinates.

PROPOSITION 4.3 ([Par1, p. 297]). *Let*

$$\rho_1((\zeta_1, v_1, t_1), (\zeta_2, v_2, t_2)) = \left||\zeta_1 - \zeta_2|^2 + |t_1 - t_2| + i(v_1 - v_2 + 2 \operatorname{Im} \zeta_1 \cdot \overline{\zeta_2})\right|^{1/2}.$$

Then ρ_1 is a distance on $\overline{\mathbf{H}^n_{\mathbf{C}}} - \{\infty\}$, invariant by the left translations and the dilatation ϕ_t is an homothety of ratio t for this metric. □

This metric defined on $\mathbf{H}^n_{\mathbf{C}}$ is an analogue of the Euclidean metric on the upper halfspace (in $\mathbf{R}^{n+1}$) of the real hyperbolic n-space. With the above, the following proposition gives a characterization of the isometric sphere in terms of our notations (cf. [Par1, Proposition 4.4]).

PROPOSITION 4.4. *Let $X \in U(q)$ such that $X\Omega \neq \Omega$. Then the isometric sphere* *is the sphere for the Cygan metric* ρ_1 *with center at* $X^{-1}(\infty)$ *and radius* $r_X = \sqrt{2/|b|}$.

Proof. By equation (3) we have

$$X^{-1}\Omega = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & A^* \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{b} \\ \bar{a} \\ -\gamma \end{pmatrix} = \bar{b} \begin{pmatrix} 1 \\ \frac{1}{2}|\zeta_0|^2 - \frac{i}{2}v_0 \\ \zeta_0 \end{pmatrix},$$

where $X^{-1}(\infty)$ has horospherical coordinates $(\zeta_0, v_0, 0)$. Let $(\zeta, v, t) \in I_X$. Choose a lift

$$Z = \begin{pmatrix} 1 \\ \frac{t}{2} + \frac{|\zeta|^2}{2} - \frac{i}{2}v \\ \zeta \end{pmatrix}$$

of (ζ, v, t). An easy computation shows that $|q(Z, \Omega)| = 1$ and that

$$|q(Z, X^{-1}\Omega)| = \frac{|b|}{2}\rho_1^2((\zeta, v, t), (\zeta_0, v_0, 0)).$$

Hence, (ζ, v, t) belongs to the sphere of radius $\sqrt{2/|b|}$ and center $X^{-1}(\infty)$. □

5. A lower bound for the volume of complex noncompact hyperbolic *n*-manifolds. This section is devoted to the proof of the following theorem.

THEOREM 5.1. *Let M be a finite-volume, complex hyperbolic manifold of dimension n with k ends. Then*

$$\mathrm{vol}(M) \geqslant \frac{k}{nI_n}$$

(see the introduction for the terminology).

Let M be a complex oriented noncompact n-manifold with finite volume. Then M is the quotient of $\mathbf{H}^n_{\mathbf{C}}$ by G, a discrete torsion-free subgroup of $\mathrm{Isom}_+(\mathbf{H}^n_{\mathbf{C}}) = \mathrm{PU}(q)$. For any group of isometries Δ of $\mathbf{H}^n_{\mathbf{C}}$, let us denote by Δ_∞ the stabilizer of ∞ in Δ. It is well known that the ends of M correspond one-to-one with the conjugacy classes of maximal parabolic subgroups of G. If ∞ is the fixed point of a parabolic element, then G_∞ preserves every horosphere ∂H_t and that $\partial H_t/G_\infty$ is compact. Furthermore, there exists $t_0 > 0$ such that the end of M corresponding to the maximal parabolic subgroup G_∞ has a neighborhood isometric to a *cusp* H_{t_0}/G_∞ (see [GaRa] for the above). In particular, $\mathrm{vol}(M)$ is bounded below by $\mathrm{vol}(H_{t_0}/G_\infty)$. A lower bound for the volume of M is obtained in two steps. In the first step, we compute the volume of these cusps. In the second one, we will use the generalized Shimizu inequality for discrete subgroups of isometries of the complex hyperbolic space to get an explicit value for t_0 at each parabolic fixed point.

For the volume computation of the cusp, we use the horospherical coordinates on the Siegel domain as in Definition 4.2. With this set of coordinates, the metric on $\mathbf{H}^n_{\mathbf{C}}$ as defined in equation (4) becomes

$$ds^2_{\mathbf{C}} = \frac{4}{t^2}\left(\left(\frac{1}{2}dv - \operatorname{Im}\zeta \cdot \overline{d\zeta}\right)^2 + \frac{1}{4}dt^2 + t d\zeta \cdot \overline{d\zeta}\right). \tag{7}$$

Let us endow the Heisenberg group with the left invariant metric defined by

$$ds^2_{\mathbf{H}} = (\tfrac{1}{2}dv - \operatorname{Im}\zeta \cdot \overline{d\zeta})^2 + d\zeta \cdot \overline{d\zeta}.$$

Let dv_ζ be the Euclidean volume form on $\mathbf{C}^{n-1} = \mathbf{R}^{2n-2}$. Then the volume form of $\mathbf{Heis}_{2n-1}$ is $d(\mathrm{vol}_H) = (1/2)\, dv\, dv_\zeta$. Note that the horospherical coordinates identify G_∞ with a discrete, torsion-free subgroup of isometries of $\mathbf{Heis}_{2n-1}$. The following lemma is the first step in getting a lower bound for the volume of the cusp.

Lemma 5.2. *Let $\tau > 0$. Then for any horoball H_τ, we have*

$$\mathrm{vol}(H_\tau/G_\infty) = \frac{2^{2n-1}}{n\tau^n}\mathrm{vol}_H(\mathbf{Heis}_{2n-1}/G_\infty). \tag{8}$$

Proof. Using equation (7), we have that the volume form in $\mathbf{H}^n_{\mathbf{C}}$ can be written as

$$d(\mathrm{vol}) = \left(\frac{2}{t}\right)^{2n}\left(\frac{dv}{2}\frac{dt}{2}\sqrt{t}^{2(n-1)}\, dv_\zeta\right) = \frac{2^{2n-1}}{t^{n+1}}\frac{dv}{2}\, dt\, dv_\zeta. \tag{9}$$

The volume element in the "slice" ∂H_τ is given by $d(\mathrm{vol})_H = (dv/2)\, dv_\zeta$. The volume of the cusp is hence given by

$$\mathrm{vol}(H_\tau/G_\infty) = \int_\tau^\infty\left(\frac{2^{2n-1}}{t^{n+1}}\, dt\right)\mathrm{vol}_H(\mathbf{Heis}_{2n-1}/G_\infty). \tag{10}$$

The assertion follows immediately. □

The following well-known theorem gives a generalization of the Bieberbach theorem to the Heisenberg group, and is in fact true for any almost flat manifold.

Theorem 5.3 ([Gro1], [BK, Theorem 1.5 (ii)]). *Let Γ be a cocompact discrete torsion-free subgroup of isometries of* $\mathbf{Heis}_{2n-1}$. *There exists a universal constant*

I_n, such that Γ contains a cocompact lattice of index less than or equal to I_n. Moreover,

$$I_n \leqslant 2(6\pi)^{(1/2)n(n-1)}.$$

We now proceed through a number of statements that will help us find canonical cusps that embed in M. Let π: $\mathbf{Heis}_{2n-1} \to \mathbf{C}^{n-1}$ be the canonical projection defined by $\pi(\zeta, v) = \zeta$.

PROPOSITION 5.4. *Let Γ be a discrete cocompact subgroup in $\mathbf{Heis}_{2n-1}$. Then $\pi(\Gamma)$ is a cocompact lattice in $\mathbf{C}^{n-1}$.*

Proof. By (6) we have

$$\ker(\pi) = \mathbf{R} \qquad \text{and} \qquad \mathbf{Heis}_{2n-1}/\mathbf{R} = \mathbf{C}^{n-1}.$$

We note that $\Gamma \cap \mathbf{R}$ is a normal subgroup of Γ, since $\mathbf{R}$ is in the center of $\mathbf{Heis}_{2n-1}$. Therefore, the group

$$G = \Gamma/(\Gamma \cap \mathbf{R}),$$

which identifies to $\pi(\Gamma)$, acts on $\mathbf{C}^{n-1}$. It is clear that G acts with bounded quotient on $\mathbf{Heis}_{2n-1}/\mathbf{R}$ and therefore that $\pi(\Gamma)$ acts cocompactly on $\mathbf{C}^{n-1}$. We now show that G acts discretely on $\mathbf{C}^{n-1}$. If not, then the orbit of 0 would accumulate on itself. That would imply the existence of $\{\gamma_n\} \in \Gamma$ and $\{t_n\} \in \mathbf{R}$ such that

$$\gamma_n(t_n)^{-1} \to \mathrm{id}.$$

Since Γ is nonabelian (it is quasi-isometric to $\mathbf{Heis}_{2n-1}$), we have that $\Gamma \cap \mathbf{R} \neq 0$. Hence,

$$\mathbf{R}/(\Gamma \cap \mathbf{R})$$

is cocompact. Therefore we can assume that $t_n \in [0, a]$ for some real number a. This will imply up to extracting a subsequence that $\gamma_n \to t \in \mathbf{R}$, which ends the proof. □

PROPOSITION 5.5. *Let $\overline{\Gamma}$ be a cocompact lattice in $\mathbf{R}^{2n}$. Let ω be the standard symplectic form on $\mathbf{C}^n = \mathbf{R}^{2n}$. Suppose that for every $x, y \in \overline{\Gamma}$, $|\omega(x, y)|$ is either 0 or $\geqslant c/4$. Then we have*

$$\operatorname{vol}(\mathbf{R}^{2n}/\overline{\Gamma}) \geqslant \frac{c^n}{2^{2n}}. \tag{11}$$

To prove this, we will need the following fact regarding symplectic forms in $\mathbf{C}^n = \mathbf{R}^{2n}$.

LEMMA 5.6. *Let θ be a symplectic form on $\mathbf{R}^{2n}$ such that $|\theta(x, y)|$ is either 0 or $\geqslant 1$ for every $x, y \in \mathbf{Z}^{2n}$. Then θ is a multiple of an integer form, and $|\det \theta| \geqslant 1$.*

Proof (J. Porti). Let $\{e_k\}_1^{2n}$ be the standard basis of $\mathbf{R}^{2n}$. For $k = 1, \ldots, 2n$, consider the linear maps $T_{e_k}: \mathbf{R}^{2n} \to \mathbf{R}$, defined by $T_{e_k}(v) = \theta(e_k, v)$. So $T_{e_k}(\mathbf{Z}^{2n})$ is a subgroup of $\mathbf{R}$ such that, by assumption, every positive element is $\geqslant 1$. Hence $T_{e_k}(\mathbf{Z}^{2n})$ is a discrete subgroup of $\mathbf{R}$, generated by a_k where $a_k \geqslant 1$. (Note that we have $T_{e_k}(\mathbf{Z}^{2n}) \neq \{0\}$, since θ is nondegenerate.)

By the antisymmetry of θ, we obtain that a_k/a_l is a rational number. This proves the first assertion.

Using the standard basis and factorizing a_k in the kth row, we note that $\det(\theta)$ can be written as

$$\det(\theta) = a_1 a_2 \cdots a_{2n} \det(W),$$

where W is an integer matrix. Since θ is nondegenerate, W is nondegenerate. Recalling that for every $k = 1, \ldots, 2n$, $a_k \geqslant 1$, the second assertion follows. □

Proof of Proposition 5.5. Up to using an homothety of ratio $\sqrt{c}/2$, we may assume $c = 4$. Let us choose $A \in \mathrm{GL}(2n, \mathbf{R})$ such that $\overline{\Gamma} = A \cdot \mathbf{Z}^n$. Set $\theta(x, y) = \omega(Ax, Ay)$. Then θ is a symplectic form which satisfies all the hypotheses of the previous lemma. We therefore have

$$\mathrm{vol}(\mathbf{R}^{2n}/\overline{\Gamma}) = \det A = \sqrt{\det \theta} \geqslant 1. \qquad \square$$

Let G be the above discrete, torsion-free subgroup of $\mathrm{PU}(q)$ of finite covolume. Of crucial importance to us is the ability to find, at each parabolic fixed point of G, a horoball whose quotient by the stabilizer of the parabolic fixed point will embed in $\mathbf{H}^n_{\mathbf{C}}/G$, disjointly from the other chosen quotients of horoballs at the inequivalent parabolic fixed points of G. This will enable us in particular to show that the lower bound on the volume of the manifold is linear with respect to the number of inequivalent parabolic fixed points, i.e., the number of cusps. The analogous construction and the disjointness property of the canonical horoballs has appeared in [Her2] for the n-dimensional real hyperbolic space.

Let p be a point in $\partial \mathbf{H}^n_{\mathbf{C}}$ fixed by a parabolic element of G. Choose $X \in \mathrm{PU}(q)$ such that $X(\infty) = p$. By Lemma 3.1 and because cocompact lattices in $\mathbf{Heis}_{2n-1}$ are nonabelian, we know that $(X^{-1}GX)_\infty$ contains nontrivial vertical translations. Define the *canonical horoball* at p to be $X(H_{|v|})$, where v is the smallest (nonzero) vertical translation in $(X^{-1}GX)_\infty$. By the computation of the stabilizer of ∞ in Section 3, this does not depend on X. Indeed, let X_0 be another element in $\mathrm{PU}(q)$ sending ∞ to p. Then $X_0^{-1}X$ fixes ∞; hence the smallest vertical translation in $(X_0^{-1}GX_0)_\infty$ is the image by $X_0^{-1}X$ of the smallest vertical translation in $(XGX^{-1})_\infty$.

PROPOSITION 5.7. *Canonical horoballs at distinct parabolic fixed points are disjoint.*

Proof. By conjugating the group and rescaling (using a dilatation) the length of the shortest (nonzero) vertical translation in the stabilizer of a parabolic fixed point, we can assume that its length equals 1 and that the fixed point is ∞. Let

$$U = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{i}{2} & 1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

be the (unique up to inverse) shortest vertical translation in G_∞.

Let $X \in \mathrm{PU}(q)$ as in equation (2) such that $X(\infty) \neq \infty$. Recall that the radius of the isometric sphere of X is $r_X = \sqrt{2/|b|}$ (see Proposition 4.4).

Following the ideas in [Her2, Theorem 2.3, Proposition 3.3], what we need to prove is that $H_1 \cap X \cdot H_1 = \varnothing$, whenever $B = XUX^{-1} \in G$. A simple calculation shows that the modulus of second entry in the first row of B equals $1/2|b|^2$. (One also needs to use the second identity of equation (14).) By the generalized Shimizu inequality (see Theorem 3.2), we must have

$$1/2|b|^2 \geqslant 2,$$

which implies that $|b| \geqslant 2$. Combining the above, we deduce that

$$r_X \leqslant 1. \tag{12}$$

By Proposition 4.4, the center of the isometric sphere of X is at $X^{-1}(\infty)$, which has $t = 0$ in its horospherical coordinates. For every $P \in H_1$, its t coordinate satisfies $t \geqslant 1$. By the definition of ρ_1 (see Proposition 4.3), it is clear that H_1 lies inside the exterior (i.e., the unbounded component) of the isometric sphere of X. As explained in the discussion following Definition 4.1, X maps the exterior of its isometric sphere I_X into the interior (i.e., the bounded component) of I_X^{-1}. The isometric sphere I_X^{-1} has the same radius as that of I_X; its center is $X(\infty)$. Hence, for every $P \in H_1$, we have

$$\rho_1(X(P), X(\infty)) \leqslant 1. \tag{13}$$

By Proposition 4.3, we must have that the t coordinate of $X(P)$ satisfies $t \leqslant 1$. We conclude that $H_1 \cap X \cdot H_1 = \varnothing$. This ends the proof. □

Remark. The last proposition gives a stronger conclusion than the one obtained by S. Kamiya [Kam2, Theorem 2.2]); see also J. Parker [Par1, Proposition 5.2]. It not only shows that the "canonical" cusps neigborhoods embed (this was precisely Kamiya's result), but furthermore that two such cusps, corresponding to distinct orbits of parabolic fixed points, are disjoint.

Proof of Theorem 5.1. By the above discussion, if k is the number of ends of $\mathbf{H}^n_{\mathbf{C}}/G$ and $p_1, \ldots, p_k$ are pairwise inequivalent representatives of all parabolic fixed points, then with H_i the canonical horoball at p_i and G_i its stabilizer in G, one has

$$\mathrm{Vol}(M) \geqslant k \min\{\mathrm{Vol}(H_i/G_i)\}.$$

Let us assume that one of the parabolic inequivalent fixed points of G is ∞. Hence we only have to find a universal lower bound for the volume of $H_{|v|}/G_\infty$, where v is the smallest (non-0) vertical translation in G_∞. Let $|v| = c > 0$. Then, by Lemma 5.2, we have

$$\mathrm{Vol}(H_c/G_\infty) \geqslant \frac{2^{2n-1}}{nc^n} \mathrm{Vol}(\mathbf{Heis}_{2n-1}/G_\infty).$$

By Theorem 5.3, we can pass to a finite-index cocompact lattice in G_∞ to obtain the following:

$$\mathrm{Vol}(H_c/G_\infty) \geqslant \frac{2^{2n-1}}{nc^n I_n} \mathrm{Vol}(\mathbf{Heis}_{2n-1}/\Gamma),$$

where $[\Gamma : G_\infty] \leqslant I_n$ and Γ is cocompact in $\mathbf{Heis}_{2n-1}$. Using the volume form of $\mathbf{Heis}_{2n-1}$, Lemma 3.1, and Proposition 5.5, we have

$$\mathrm{Vol}(\mathbf{Heis}_{2n-1}/\Gamma) = \frac{c}{2} \mathrm{Vol}(\mathbf{C}^{n-1}/\pi(\Gamma)) \geqslant \frac{c^n}{2^{2n-1}}.$$

We combine the above inequalities to obtain the final result

$$\mathrm{Vol}(M) \geqslant \frac{k}{nI_n}. \qquad \square$$

For $n = 2$, we can give the precise computation of the best constant I_n. Note that in that case, the standard symplectic form ω on $\mathbf{R}^2$ is precisely the area form of $\mathbf{R}^2$.

PROPOSITION 5.8. *We have $I_2 = 6$.*

Proof. This proof may be found between the lines in [ShSt]. Note that $\mathrm{Nil} = \mathbf{Heis}_3$ fibers over $\mathbf{R}^2$ with fiber $\mathbf{R}$, that the isometries of Nil preserve this fibration, and that the projection of an isometry of Nil is an isometry of $\mathbf{R}^2$. Hence (see [Sco2]) the closed 3-manifold $M = \mathrm{Nil}/\Gamma$ is a Seifert fibered space with basis a Euclidean 2-orbifold E. Since Nil contains no orientation reversing isometry (see [Sco2]), the underlying topological space of E is a closed surface, and the points having nontrivial stabilizers are isolated.

Let $\tilde{E} \to E$ be a finite orbifold covering (of smallest degree) with $\tilde{E}$ a closed flat surface, and let $\tilde{M} \to M$ be the induced covering. Note that $\pi_1 \tilde{M}$ (which is a subgroup of $\pi_1 M$) is now contained in Nil. One has the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \pi_1 \tilde{M} & \longrightarrow & \pi_1 \tilde{E} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \Gamma & \longrightarrow & \pi_1^{\mathrm{orb}}(E) & \longrightarrow & 1
\end{array}$$

Denote by $S(n_1, \ldots, n_k)$ the 2-orbifold of underlying topological space a surface S, with k singular points having cyclic stabilizers of orders n_i. In the case of a Euclidean 2-orbifold, the flat metric has at such a point a conical singularity of angle $2\pi/n_i$. (See the classification of Euclidean 2-orbifolds, for example, in [Thu, Section 5.5, Table 2].) Hence if E is orientable, E is the torus, $\mathbf{S}^2(2,2,2,2), \mathbf{S}^2(2,4,4)$, $\mathbf{S}^2(3,3,3)$ or $\mathbf{S}^2(2,3,6)$. If E is nonorientable, then E is the Klein bottle or $\mathbf{P}^2(2,2)$. Hence if Q is the quotient $\Gamma/\pi_1 \tilde{M} \simeq \pi_1^{\mathrm{orb}}(E)/\pi_1 \tilde{E}$, then a case-by-case study shows that the order of Q is $1,2,4,3,6,1,2$ respectively; hence less than 6 (attained by $\mathbf{S}^2(2,3,6)$). □

Remark. A. Szscepanski told us that the computation of I_3 might be possible using the list of 4-dimensional Euclidean orbifolds in [BBNWZ].

Appendix

A generalization of the Shimizu inequality for the complex hyperbolic space. We found the following discreteness criterion independently of J. Parker (see [Par3]) to which we refer for applications. Since we are using a different hermitian form q, our proof is technically much simpler. Let

$$X_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}.$$

Proposition A.1. *Let G be a discrete subgroup of $U(q)$. Suppose that G contains $X_{s,\zeta}$ with $(s,\zeta) \neq (0,0)$. Then for every $X \in G - G_\infty$, we have (with the notations of equation (2))*

$$\sup\{|b|, |\beta|, |\gamma|, |A - I|\} \geqslant \frac{1 - 2|\zeta|}{|s| + 2|\zeta|}.$$

Note that this statement gives new information only if $|\zeta| \leqslant 1/2$. J. Parker observed that since this inequality is not homogeneous, conjugating the group G by a dilatation

$$\begin{pmatrix} k & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & I \end{pmatrix}$$

gives such an inequality for all k (note that b is changed in $k^2 b$, β in $k\beta$, γ in $k\gamma$, s in $(1/k^2)s$ and ζ in $(1/k)\zeta$).

Proof. Since $XX^{-1} = I$ (I being the identity matrix of the corresponding dimension), one obtains the following set of identities.

$$(14)\qquad \begin{cases} a\bar{d} + b\bar{c} - \gamma^*\delta = 1 \\ a\bar{b} + b\bar{a} - \gamma^*\gamma = 0 \\ c\bar{d} + d\bar{c} - \delta^*\delta = 0 \\ \bar{a}\beta + \bar{b}\alpha - A\gamma = 0 \\ \bar{c}\beta + \bar{d}\alpha - A\delta = 0 \\ \alpha\beta^* + \beta\alpha^* - AA^* = -I \end{cases}$$

Define by induction $X_0 = X$, and $X_{n+1} = X_n X_{s,\zeta} X_n{}^{-1}$. By a straightforward computation and (14), we obtain the following relations.

$$(15)\qquad \begin{cases} a_{n+1} = sb_n\overline{d_n} + \overline{d_n}\gamma_n^*\zeta - b_n\zeta^*\delta_n + 1 \\ b_{n+1} = s|b_n|^2 + 2\,\mathrm{Im}\,\overline{b_n}\gamma_n^*\zeta \\ c_{n+1} = s|d_n|^2 + 2\,\mathrm{Im}\,\overline{d_n}\delta_n^*\zeta \\ d_{n+1} = s\overline{b_n}d_n + \overline{b_n}\delta_n^*\zeta - d_n\zeta^*\gamma_n + 1 \\ \gamma_{n+1}^* = -sb_n\beta_n^* + b_n\zeta^*A_n^* - \gamma_n^*\zeta\beta_n^* \\ \delta_{n+1}^* = -sd_n\beta_n^* + d_n\zeta^*A_n^* - \delta_n^*\zeta\beta_n^* \\ \alpha_{n+1} = s\overline{d_n}\beta_n + \zeta^*\delta_n\beta_n + \overline{d_n}A_n\zeta \\ \beta_{n+1} = s\overline{b_n}\beta_n + \zeta^*\gamma_n\beta_n + \overline{b_n}A_n\zeta \\ A_{n+1} = -s\beta_n\beta_n^* + \beta_n\zeta^*A_n^* - A_n\zeta\beta_n^* + I \end{cases}$$

Define $t_n = \sup\{|b_n|, |\beta_n|, |\gamma_n|, |A_n - I|\}$ and $\mu = |s| + 2|\zeta| \neq 0$. By (15) we have

$$\begin{cases} |b_{n+1}| \leqslant \mu {t_n}^2 \\ |\gamma_{n+1}| \leqslant \mu {t_n}^2 + |b_n||\zeta| \\ |\beta_{n+1}| \leqslant \mu {t_n}^2 + |b_n||\zeta| \\ |A_{n+1} - I| \leqslant \mu {t_n}^2 + 2|\beta_n||\zeta| . \end{cases}$$

(Note that if $\zeta = 0$, then we directly have $|b_{n+1}| \leqslant |s||b_n|^2$, which proves Kamiya's result 3.2.) We therefore have

$$t_{n+1} \leqslant \mu {t_n}^2 + 2|\zeta| t_n . \tag{16}$$

Suppose that $t_0 < (1 - 2|\zeta|)/\mu$. Choose $0 < \varepsilon < 1$ (close to one) such that

$$\mu t_0 + 2|\zeta| \leqslant \varepsilon .$$

By induction, it follows from (16) and the above equation that

$$t_n \leqslant \varepsilon^n t_0 .$$

Therefore t_n tends to 0. If n is big enough, then $|A_n| \leqslant 2$, and by (15), we have

$$|d_{n+1} - 1| \leqslant 3t_n\left(s + |\zeta| + |\zeta|^2\right) \sup\{|d_n - 1|, |\delta_n - \zeta|, 1\}$$

and that

$$|\delta_{n+1} - \zeta| \leqslant 3t_n\left(s + |\zeta| + |\zeta|^2|\right) \sup\{|d_n - 1|, |\delta_n - \zeta|, 1\} + |\zeta||d_n - 1| .$$

Hence d_n converges to 1, and δ_n converges to ζ. By a similar argument, a_n converges to 1, c_n converges to s, and α_n converges to ζ. If $t_0 \neq 0$, this contradicts the assumption that the group generated by X_0 and $X_{s,\zeta}$ is discrete. But if $t_0 = 0$, then X_0 fixes the point at infinity and hence belongs to G_∞. Hence we must have

$$t_0 \geqslant \frac{1 - 2|\zeta|}{\mu} ,$$

which ends the proof. □

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