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Hausdorff dimension of diophantine geodesics in negatively curved manifolds

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Abstract. We provide a sharp estimate for the visual dimension of the set of geodesic rays, starting from any fixed point p in a closed pinched negatively curved Riemannian manifold, that are coming back exponentially close to p infinitely often.

1. Introduction

Let M be a smooth complete Riemannian manifold of dimension $n \geqq 2$ with pinched negative curvature $-a^2 \leqq K \leqq -1$, with $1 \leqq a < \infty$. Let h be the volume entropy of $\tilde{M}$, that is

$$h = \limsup_{R \to +\infty} \frac{\log \operatorname{vol} B_{\tilde{M}}(\tilde{p}, R)}{R},$$

where $\tilde{M} \to M$ is a universal cover of M and $\tilde{p}$ any point in $\tilde{M}$. Fix two points p, q in M, and endow the unit tangent sphere at p with Gromov's visual metric (the definition is recalled in section 2) and with the Hausdorff measures defined by this metric.

In this paper, we provide sharp estimates on the Hausdorff dimension of the set of geodesic rays γ starting from p that accumulate on q exponentially fast. More precisely:

Definition 1.1. Let $\alpha \in [0, +\infty[$, a geodesic γ starting from p is *α-Liouville* at q if there exist a constant $K > 0$ and a sequence $(t_n)_{n \in \mathbb{N}}$ converging to $+\infty$ such that, for every n in $\mathbb{N}$,

$$d\big(q, \gamma(t_n)\big) \leqq K e^{-\alpha t_n}.$$

If $\alpha = 0$, then the Liouville geodesic rays are exactly the recurrent geodesic rays.

Theorem 1.2. *If M is compact, then the Hausdorff dimension D_α of the set of α-Liouville geodesic rays starting from a given point in M satisfies*

$$\frac{h}{1+\alpha} \leqq D_\alpha \leqq \frac{h}{1+\frac{\alpha}{a}}.$$

Note that the bounds depend neither on p nor on q. If the curvature is constant and equals -1, then the visual metric on the unit tangent sphere coincides with the spherical metric, and the Hausdorff dimension of the set of α-Liouville geodesic rays starting from a given point in M is exactly $\frac{n-1}{1+\alpha}$.

No smoothness assumption is necessary, this result also holds when M is a metric space with curvature bounded above by -1 and below by $-a^2$ in the sense of Alexandrov (see for example [GH]). For the assertion regarding the upper bound, the compactness assumption of M can be removed, by replacing h with the critical exponent δ of the covering group $\tilde{M} \to M$ (see Theorem 4.1).

When $\alpha = 0$, and without the compactness assumption, this result is due (besides partial cases by S. Dani, C. Aravinda, J. Fernández-M. Melián, C. Aravinda-E. Leuzinger and B. Stratmann) to Bishop-Jones [BJ] for rank one symmetric spaces of non compact type, and to [Pau] for hyperbolic metric spaces in the sense of Gromov, again up to replacing h by δ. When $\alpha > 0$, this result is a contribution to the Hill-Velany's [HV1] program of "shrinking targets", in the case (that they did not develop) of the geodesic flow of negatively curved Riemannian manifolds. Also note that the paper [HV2] gives, in constant curvature, an analogous result when the point q is "at infinity" (and a parabolic fixed point). We plan to extend it to our variable curvature setting, see [HP2].

Definition 1.3. A geodesic ray γ starting from p is *α-Diophantine* if there exists a constant $K > 0$ such that for all t in $[1, +\infty[$,

$$d(q, \gamma(t)) \geqq K e^{-\alpha t}.$$

We say that γ is *of Roth type* if it is α-Diophantine for every $\alpha > 0$. We prove that almost every (for the Hausdorff measure of the visual sphere) geodesic ray starting from p badly approximates q, in the sense of the next result, which immediately follows from Theorem 1.2, since the Hausdorff dimension of $T_p^1 M$ is h.

Corollary 1.4. *Almost every geodesic ray starting from a given point in M is of Roth type.*

These results for the geodesic flow of a negatively curved manifold are analogous to results in metric diophantine approximation theory (see for example [Khi]). We make this analogy explicit in section 2 after recalling the basic definitions that are needed for the proofs. See also [HP], [KM], [BD] for other connections. The (easier) upper bound is proved in section 4 and the lower bound in section 5. We also consider (see Theorem 2.1) the case when the function $t \mapsto e^{-\alpha t}$ in the definition of α-Liouville geodesic rays is replaced by $t \mapsto g(t)$ with $g(t) \to 0$ as t goes to $+\infty$. To obtain our bounds, we give new estimates on the size of the shadows of tiny balls on the boundary of $\tilde{M}$, that were known for instance by Patterson-Sullivan when the radius of the ball is fixed (see section 3).

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2. Background and notation

We recall some notations, definitions and results of [Bou], [GH] about negatively curved metric space, that have essentially been introduced by M. Gromov.

Let X be a proper CAT(-1) space (for example a simply connected complete Riemannian manifold $\tilde{M}$ with curvature $K \leqq -1$), and let Γ be a discrete group of isometries of X (for example the covering group Γ_M of a universal cover $\tilde{M} \to M$ with M as in the introduction). Let x, y be points in X, with x considered as the base point.

The *boundary* ∂X of X is the space of all geodesic rays in X, where two rays are identified if they remain within bounded Hausdorff distance. We call Γ *non elementary* if no finite index subgroup has a global fixed point in $X \cup \partial X$. The *Poincaré series* of Γ is

$$P(s) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}.$$

This series converges (independently of x, y) if $s > \delta$ and diverges if $s < \delta$ with

$$\delta = \limsup_{R \to +\infty} \frac{1}{R} \log \operatorname{Card}\bigl(\Gamma y \cap B_X(x, R)\bigr).$$

We will always assume that δ is positive and finite. This is for example the case when Γ is non elementary and X also has curvature bounded from below by $-a^2$ in the Alexandrov's sense. Note that if $\Gamma = \Gamma_M$ with M as in the introduction and if M is compact, then δ is the volumic entropy h defined in the introduction.

Let $a, b \in \partial X$. Their Gromov product with respect to the base point x in X is defined by

$$(a, b)_x = \lim_{t \to +\infty} \frac{1}{2}\Bigl(d\bigl(x, a(t)\bigr) + d\bigl(x, b(t)\bigr) - d\bigl(a(t), b(t)\bigr)\Bigr)$$

independently of the geodesic rays a, b: $[0, +\infty[\to X$ representing a, b. The *visual distance* d_x on ∂X is then defined by

$$d_x(a, b) = \begin{cases} 0 & \text{if } a = b, \\ e^{-(a, b)_x} & \text{otherwise.} \end{cases}$$

We will denote by $\mathscr{B}(\xi, r) = \mathscr{B}_x(\xi, r)$ the open ball of center ξ and radius $r > 0$ in ∂X endowed with the visual distance d_x. An isometry γ of X extends to a homeomorphism of ∂X which is an isometry between d_x and $d_{\gamma x}$. If M is as in the introduction, and p is a point in M, then the unit tangent sphere $T^1_{\tilde{p}} \tilde{M}$ at any lift $\tilde{p}$ of p in a universal cover $\tilde{M} \to M$ is homeomorphic to the sphere at infinity by the map which associates to a unit tangent vector v at $\tilde{p}$ the point at infinity of the unique geodesic c starting from $\tilde{p}$ with $c'(0) = v$. By equivariance of the visual distances with respect to the isometries, the pull back of the visual distance $d_{\tilde{p}}$ from ∂M to $T^1_{\tilde{p}} \tilde{M}$ projects to the visual distance on $T^1_p M$ used in the introduction for the statement of Theorem 1.2.

For $s > 0$, one defines the s-dimensional *Hausdorff measures* μ_s on ∂X associated to the visual metric d_x, as follows. For $\eta > 0$ and A a subset of ∂X, let $\mu_{s,\eta}(A) = \inf \sum_i r_i^s$, where the infimum is taken over all countable covers of A by balls of radius $r_i \leqq \eta$. Define

$$\mu_s(A) = \lim_{\eta \to 0} \mu_{s,\eta}(A).$$

The limit exists, and there is a unique $\sigma \in [0, +\infty]$ such that $\mu_s(A) = +\infty$ if $0 \leqq s < \sigma$ and $\mu_s(A) = 0$ if $\sigma < s$, which is called the *Hausdorff dimension* of A (see [Fal], Section 1.2).

If (E, d) is a metric space, and B is a ball of radius $r > 0$, for every $\lambda > 0$, we will denote by λB the ball of radius λr and same center.

The *shadow* $\mathcal{O}A = \mathcal{O}_x A$ of a subset A of X seen from x is the set of points ξ in ∂X such that the (unique) geodesic ray from x to ξ has a non-empty intersection with A. The *cone* $\mathcal{C}A = \mathcal{C}_x A$ based at x over a subset A of ∂X is the union of the images of the geodesic rays starting from x with endpoints in A. The *shadow cone* $\mathcal{C}\mathcal{O}A$ of a subset A of X seen from x is the cone based at x over the shadow seen from x of A.

If $\alpha > 0$, one can take $K = 1$ in the definition of the α-Liouville geodesic rays. Any geodesic ray passing through q infinitely many times (a periodic one, for example) is α-Liouville, but there are at most countably many of them (since $\pi_1 M$ is countable).

Let $\pi: X \to X/\Gamma$ be the canonical projection. Given $g: [0, +\infty[\to]0, +\infty[$ with $g(t)$ converging to 0 as t goes to $+\infty$, say that a geodesic ray φ starting from x is *g-Liouville* if there exists a sequence $(t_n)_{n \in \mathbb{N}}$ converging to $+\infty$ such that, for every n in $\mathbb{N}$,

$$d_{X/\Gamma}\big(\pi(y), \pi \circ \varphi(t_n)\big) \leqq g(t_n).$$

Theorem 1.2 in the introduction follows from the following theorem, with $p = \pi(x)$, $q = \pi(y)$ and $g(t) = e^{-\alpha t}$.

Theorem 2.1. *Let X be a smooth complete simply connected Riemannian manifold of dimension $n \geqq 2$, with pinched negative curvature $-\infty < -a^2 \leqq K \leqq -1$, with $a \geqq 1$, and let Γ be a non elementary discrete group of isometries of X.*

(1) *If $\alpha = \liminf_{t \to \infty} \dfrac{-\log g(t)}{t}$, then the Hausdorff dimension of the set of g-Liouville geodesic rays is at most $\dfrac{\delta}{1 + \frac{\alpha}{a}}$.*

(2) *If $\beta = \limsup_{t \to \infty} \dfrac{-\log g(t)}{t}$ and if Γ is cocompact, then the Hausdorff dimension of the set of g-Liouville geodesic rays is at least $\dfrac{\delta}{1 + \beta}$.*

An analog of this statement in the constant curvature case was obtained in [Vel]. The above theorem follows from Theorem 4.1 and Theorem 5.1 (below) by the following crucial lemma (where $\bar{B}$ denotes a closed ball):

Lemma 2.2. *A geodesic ray φ starting from x is g-Liouville if its point at infinity $\varphi(\infty)$ belongs to infinitely many shadows $\mathcal{O}\bar{B}\left(\gamma y, \frac{1}{2}g\big(d(x,\gamma y)\big)\right)$, and only if $\varphi(\infty)$ belongs to infinitely many shadows $\mathcal{O}\bar{B}\big(\gamma y, 2g\big(d(x,\gamma y)\big)\big)$, where γ runs over Γ.*

Proof. Assume first that $\varphi(\infty)$ belongs to infinitely many shadows

$$\mathcal{O}\bar{B}\left(\gamma y, \frac{1}{2}g\big(d(x,\gamma y)\big)\right).$$

Then the geodesic ray φ meets the balls, centered at orbit points $\gamma_n y$ and having radius $r_n = \frac{1}{2}g\big(d(x,\gamma_n y)\big)$, such that $t_n = d(x,\gamma_n y) \to +\infty$. Let p_n be the orthogonal projection of $\gamma_n y$ on the image of φ, so that in particular $d(\gamma_n y, p_n) \leqq \frac{1}{2}g(t_n)$. Then since $g(t) \to 0$ as $t \to +\infty$, one has (for n big enough):

$$\begin{aligned} d\big(\pi(y), \pi\circ\varphi(t_n)\big) &= d\big(\gamma_n y, \varphi(t_n)\big) \leqq d(\gamma_n y, p_n) + d\big(p_n, \varphi(t_n)\big) \\ &\leqq d(\gamma_n y, p_n) + \big(d\big(\varphi(t_n), x\big) - d(x, p_n)\big) = d(\gamma_n y, p_n) + \big(d(\gamma_n y, x) - d(x, p_n)\big) \\ &\leqq 2d(\gamma_n y, p_n) \leqq g(t_n). \end{aligned}$$

The other direction follows by a similar argument. □

We now explain the analogy with classical results in metric diophantine approximation theory (see [Khi]). Recall that an irrational real number z satisfies *a Liouville condition of order* $\alpha \geqq 0$ if there exists a constant $K > 0$ and infinitely many reduced rational numbers $\frac{p_n}{q_n}$ with $\left|z - \frac{p_n}{q_n}\right| \leqq \frac{K}{q_n^{2+\alpha}}$. In particular every irrational real number satisfies a Liouville condition of order 0. A real number z satisfies *a diophantine condition of order* $\alpha \geqq 0$ if there exists a constant $K > 0$ such that for every reduced rational number $\frac{p}{q}$, one has $\left|z - \frac{p}{q}\right| \geqq \frac{K}{q^{2+\alpha}}$. A real number z is *of Roth type* if it satisfies a diophantine condition of order α for every $\alpha > 0$. It is well known (see [Khi]) that almost every (in the sense of the Lebesgue measure) real number is of Roth type, and our Corollary 1.4 is analogous to this result. Similarly, the upperbound in Theorem 1.2 is related to Dodson's result in [Dod]. But the connection with the classical metric diophantine theory can be made even sharper, as follows.

Let $\mathbb{T}^2$ be the quotient of $\mathbb{R}^2$ by its standard integer lattice $\mathbb{Z}^2$, endowed with its flat metric, and π: $\mathbb{R}^2 \to \mathbb{T}^2$ be the standard projection. Let O denote the projection of the zero of $\mathbb{R}^2$ in $\mathbb{T}^2$. For every real number z, let γ_z be one of the two geodesic rays in $\mathbb{T}^2$ starting from O, which is the projection of the half line L_z of $\mathbb{R}^2$ starting from zero with slope z. The following fact explains the relationship between the definitions in the introduction and the classical notions of diophantine approximation theory.

Proposition 2.3. *The real number z satisfies a Liouville condition of order $\alpha \geqq 0$ if and only if there exist a constant $K > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$ such that, for every n in $\mathbb{N}$,*

$$d_{\mathbb{T}^2}\big(O, \gamma_z(t_n)\big) \leqq K t_n^{-(1+\alpha)}.$$

Proof. The point of coordinates (q, qz) on the half-line L_z is at distance $|qz - p| < 1$ from (one of) the closest integer(s) point (q, p), and at distance $q\sqrt{1+z^2}$ from the origin. Hence the distance from $\gamma_z(q\sqrt{1+z^2})$ to O in $\mathbb{T}^2$ is $|qz - p|$. This proves the only if part. The converse direction is as easy. $\square$

The fact that exponentials are replaced by powers in this proposition is due to the fact that the torus $\mathbb{T}^2$ is flat and not negatively curved.

3. A consequence of the exponential divergence of geodesics

In this section we prove a technical lemma which will be used later on. Let X be a smooth complete simply connected Riemannian manifold, with pinched negative curvature $-\infty < -a^2 \leqq K \leqq -1$, where $a \geqq 1$. Fix a point $x \in X$. For $z \neq x$, let $t \mapsto z_t$ be the (unit speed) geodesic ray starting from x and passing through z. Let z_∞ be its point at infinity.

For every $\varepsilon > 0$ and metric space Y with distance d, we denote by εY the set Y endowed with the metric $d_\varepsilon = \varepsilon d$.

Let $B \in X$ be a ball of a given radius. It is well known that (for all CAT(-1) spaces) the shadow of B is comparable to a visual ball of radius which equals the exponential of minus the distance between the base point and the center of the ball B. But we need more precise estimates for the case when the radius of the ball B converges to 0. For these estimates, we need the hypothesis of pinched curvature.

Lemma 3.1. *There exist positive constants c_1, c_2, c_3 with c_1 universal and c_2, c_3 depending only on a, such that for all z in X with $d(x,z) \geqq c_2$, for all $R > 0$ with $R \leqq c_3$ and $R \leqq d(x,z)$ then*

$$\mathscr{B}(z_\infty, Re^{-d(x,z)}) \subset \mathscr{O}_x\big(B(z,R)\big) \subset \mathscr{B}(z_\infty, c_1 R^{\frac{1}{a}} e^{-d(x,z)}).$$

Proof. Let $c_1 = e^{\frac{8}{e}}$ and $c_2 = \dfrac{\log 2}{2a}$. Since $\sinh'(0) = 1$, there exists a universal constant $c_3' > 0$ such that $\sinh t < 2t$ if $0 \leqq t \leqq c_3'$. Let $c_3 = \dfrac{c_3'}{a}$.

Let us first prove the right inclusion. Let r: $[0, +\infty[\to X$ be a geodesic ray starting at x, such that the orthogonal projection p of z onto the image of r satisfies $d(z,p) < R$. Let θ be the angle at x between the tangent vectors of $t \mapsto z_t$ and $t \mapsto r(t)$. We have to prove that the point at infinity $r(\infty)$ of r is at visual distance less than $c_1 R^{\frac{1}{a}} e^{-d(x,z)}$ from z_∞.

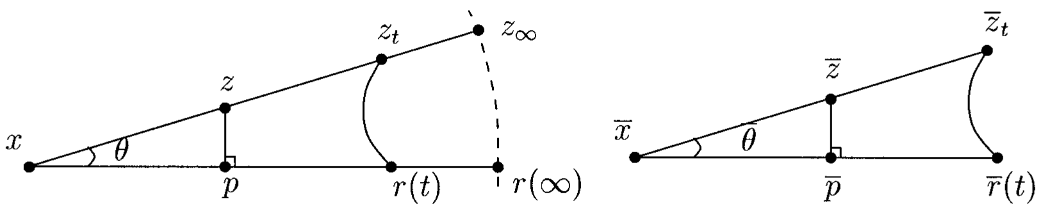

Figure 1. Comparing shadows and visual balls.

Let $(\bar{x}, \bar{z}_t, \bar{r}(t))$ be a comparison triangle in $\frac{1}{a}\mathbb{H}^2$ (whose distance we denote by $d_{\frac{1}{a}}$) of the geodesic triangle $(x, z_t, r(t))$, with $\bar{\theta}$ its angle at $\bar{x}$, $\bar{z}$ the point corresponding to z, $\bar{p}$ the orthogonal projection of $\bar{z}$ onto the opposite side $[\bar{x}, \bar{r}(t)]$. Since $\frac{1}{a}\mathbb{H}^2$ has constant curvature $-a^2$, the curvature of X is greater than or equal to the curvature of $\frac{1}{a}\mathbb{H}^2$, hence

$$d_{\frac{1}{a}}(\bar{z}, \bar{p}) \leqq d(z, p) \quad \text{and} \quad \bar{\theta} \leqq \theta.$$

By the hyperbolic sine rule in $\frac{1}{a}\mathbb{H}^2$,

$$\sin\frac{\bar{\theta}}{2} = \frac{\sinh\frac{a}{2}d_{\frac{1}{a}}(\bar{z}_t, \bar{r}(t))}{\sinh ad_{\frac{1}{a}}(\bar{x}, \bar{z}_t)} \quad \text{and} \quad \sin\bar{\theta} = \frac{\sinh ad_{\frac{1}{a}}(\bar{z}, \bar{p})}{\sinh ad_{\frac{1}{a}}(\bar{x}, \bar{z})}.$$

Since $d_{\frac{1}{a}}(\bar{x}, \bar{z}_t) = d(x, z_t) = t$ and $d_{\frac{1}{a}}(\bar{z}_t, \bar{r}(t)) = d(z_t, r(t))$, one has

$$\lim_{t\to+\infty} e^{-\frac{1}{2}(2t - d(z_t, r(t)))} = \lim_{t\to+\infty}\left(\frac{e^{\frac{a}{2}d_{\frac{1}{a}}(\bar{z}_t, \bar{r}(t))}}{e^{ad_{\frac{1}{a}}(\bar{x}, \bar{z}_t)}}\right)^{\frac{1}{a}} = \lim_{t\to+\infty}\left(\sin\frac{\bar{\theta}}{2}\right)^{\frac{1}{a}}.$$

Since $d(z, p) < R \leqq d(x, z)$, one has $\theta \leqq \frac{\pi}{2}$, hence $\bar{\theta} \leqq \frac{\pi}{2}$, so $\sin\frac{\bar{\theta}}{2} \leqq \sin\bar{\theta}$, and

$$d_x(z_\infty, r(\infty)) \leqq \left(\frac{\sinh ad_{\frac{1}{a}}(\bar{z}, \bar{p})}{\sinh ad_{\frac{1}{a}}(\bar{x}, \bar{z})}\right)^{\frac{1}{a}}.$$

Since $aR \leqq c_3'$, one has $\sinh ad_{\frac{1}{a}}(\bar{z}, \bar{p}) \leqq \sinh ad(z, p) < \sinh aR \leqq 2aR$. Since

$$ad(x, z) \geqq ac_2 = \frac{\log 2}{2},$$

one has $\sinh ad(x, z) \geqq \frac{1}{4}e^{ad(x,z)}$. Hence

$$d_x(z_\infty, r(\infty)) < (8a)^{\frac{1}{a}} R^{\frac{1}{a}} e^{-d(x,z)}.$$

Therefore, the right inclusion in Lemma 3.1 holds since $(8a)^{\frac{1}{a}} \leqq c_1$.

Remark 3.2. If one drops the assumption $R \leqq c_3$, one gets

$$\mathcal{O}_x(B(z, R)) \subset \mathcal{B}(z_\infty, (4\sinh(aR))^{\frac{1}{a}} e^{-d(x,z)}).$$

We now prove the left inclusion in Lemma 3.1. Let ξ be a point in $\mathcal{B}(z_\infty, Re^{-d(x,z)})$, let r: $[0, +\infty[\to X$ be the geodesic ray starting at x whose point at infinity is $r(\infty) = \xi$. Let p be the orthogonal projection of z onto the image of r. We only have to prove that $d(z, p) < R$.

Let $(\bar{x}, \bar{z}_t, \bar{r}(t))$ be a comparison triangle in $\mathbb{H}^2$ (whose distance we denote by d_1) of the geodesic triangle $(x, z_t, r(t))$, with $\bar{\theta}$ its angle at $\bar{x}$. Let $\bar{z}$ be the point corresponding to z, and $\bar{p}$ be the orthogonal projection of $\bar{z}$ onto the opposite side $[\bar{x}, \bar{r}(t)]$. Since the curvature of X is less than or equal to the curvature of $\mathbb{H}^2$, one has

$$d(z, p) \leqq d_1(\bar{z}, \bar{p}).$$

By the hyperbolic sine rule in $\mathbb{H}^2$,

$$\sin\frac{\bar{\theta}}{2} = \frac{\sinh\frac{1}{2}d_1(\bar{z}_t, \bar{r}(t))}{\sinh d_1(\bar{x}, \bar{z}_t)} \quad \text{and} \quad \sin\bar{\theta} = \frac{\sinh d_1(\bar{z}, \bar{p})}{\sinh d_1(\bar{x}, \bar{z})}.$$

Since $\bar{\theta} \in [0, \pi]$, one has

$$d(z, p) \leqq \frac{1}{2}e^{d(x,z)}\frac{\sinh d(z, p)}{\sinh d(x, z)} \leqq \frac{1}{2}e^{d(x,z)}\sin\bar{\theta} \leqq e^{d(x,z)}\sin\frac{\bar{\theta}}{2}.$$

The right hand side converges, as t goes to $+\infty$, to

$$e^{d(x,z)}d_x(z_\infty, r(\infty)) < e^{d(x,z)}Re^{-d(x,z)} = R.$$

Therefore, the left inclusion in Lemma 3.1 holds. $\square$

4. An upper bound for the Hausdorff dimension

Let X be a smooth complete simply connected Riemannian manifold of dimension $n \geqq 2$, with pinched negative curvature $-\infty < -a^2 \leqq K \leqq -1$, with $a \geqq 1$. Let $x, y \in X$ and $f\colon [0, +\infty[\to \mathbb{R}$ be any function with $f(t)$ converging to $+\infty$ as t goes to $+\infty$. For $z \neq x$, let $t \mapsto z_t$ be the (unit speed) geodesic ray starting from x and passing through z, and let z_∞ be its point at infinity. Let Γ be a non elementary discrete group of isometries of X. For every z in X, define B_z to be the closed ball centered at z of radius $r_z = e^{-f(d(x,z))}$ and $\mathcal{O}_z = \mathcal{O}B_z$ the shadow of B_z seen from x. Let $\mathcal{O}_f$ be the set of points ξ in ∂X that belongs to infinitely many shadows $\mathcal{O}_{\gamma y}$, i.e. such that there exist infinitely many γ in Γ with $\xi \in \mathcal{O}_{\gamma y}$.

Theorem 4.1. *If* $\alpha = \liminf_{t\to+\infty} \frac{f(t)}{t}$, *then* $\dim_{\text{Haus}}(\mathcal{O}_f) \leqq \frac{\delta}{1 + \frac{\alpha}{a}}$.

Proof. First let us assume that $\alpha < +\infty$. It is sufficient to prove that for every $s > \frac{\delta}{1 + \frac{\alpha}{a}}$, the s-dimensional Hausdorff measure $\mu_s(\mathcal{O}_f)$ is finite. Fix such an s.

Since $f(t)$ tends to $+\infty$ as t goes to $+\infty$, by Lemma 3.1 (which required $d(x, z) > R$), there exists a finite subset P of Γ such that for every γ in $\Gamma - P$, one has:

$$\mathcal{O}_{\gamma y} \subset \mathcal{B}\left((\gamma y)_\infty, c_1(r_{\gamma y})^{\frac{1}{a}}e^{-d(x,\gamma y)}\right).$$

Fix $\eta > 0$ such that $\mu_s(\mathcal{O}_f) \leqq \mu_{s,\eta}(\mathcal{O}_f) + 1$. Since $f(t)$ is positive for t big enough, there exists a finite subset P' of Γ, containing P, such that

$$c_1 (r_{\gamma y})^{\frac{1}{a}} e^{-d(x,\gamma y)} \leqq \eta$$

for every γ in $\Gamma - P'$. Let $\varepsilon > 0$ be such that $\varepsilon < 1 + \frac{\alpha}{a} - \frac{\delta}{s}$, which exists by our choice of s. Since $\alpha = \liminf_{t\to+\infty} \frac{f(t)}{t}$, there exists $T > 0$ such that if $t \geqq T$, then $f(t) \geqq (\alpha - \varepsilon)t$. Let P'' be a finite subset of Γ, containing P', such that $d(x, \gamma y) \geqq T$ for every γ in $\Gamma - P''$. Since $(\mathcal{O}_{\gamma y})_{\gamma\in\Gamma-P''}$ is a covering of $\mathcal{O}_f$, one has

$$\mu_{s,\eta}(\mathcal{O}_f) \leqq \sum_{\gamma\in\Gamma-P''} c_1^s (r_{\gamma y})^{\frac{s}{a}} e^{-sd(x,\gamma y)} \leqq c_1^s \sum_{\gamma\in\Gamma-P''} e^{-s(1+\frac{\alpha-\varepsilon}{a})d(x,\gamma y)}.$$

Since $s\left(1 + \frac{\alpha-\varepsilon}{a}\right) > \delta$, this last series converges, and the result is proved.

If $\alpha = +\infty$, then we formally replace α in the proof above by any $A > 0$, and we get that $\dim_{\mathrm{Haus}}(\mathcal{O}_f) \leqq \dfrac{\delta}{1 + \frac{A}{a}}$. Letting A tend to $+\infty$, it follows that $\dim_{\mathrm{Haus}}(\mathcal{O}_f) = 0$, as wanted. $\square$

5. A lower bound for the Hausdorff dimension

We keep the same notation as in the previous section. We want to prove the following result.

Theorem 5.1. *If Γ is cocompact and if $\beta = \limsup_{t\to+\infty} \frac{f(t)}{t}$, then* $\dim_{\mathrm{Haus}}(\mathcal{O}_f) \geqq \dfrac{\delta}{1+\beta}$.

Fix $\varepsilon > 0$ with $s = \delta - 2\varepsilon > 0$. There is nothing to prove if $\beta = +\infty$, therefore we assume that β is finite (note that $\beta \geqq 0$ since $f(t)$ is eventually positive).

Since there exists $T \geqq 0$ such that $f(t) \leqq (\beta + 1)t$ for all $t \geqq T$, up to replacing $f(t)$ by $\bar{f}(t) = \sup_{t'\in[T,t]} f(t')$ for $t \geqq T$, which satisfies $\mathcal{O}_{\bar{f}} \subset \mathcal{O}_f$ and $\limsup_{t\to+\infty} \frac{\bar{f}(t)}{t} = \beta$, we may assume that f is nondecreasing on $[T, +\infty[$. Since the validity of the result is unchanged if we modify f on a compact subset of $[0, +\infty[$, and since $f(t)$ tends to $+\infty$ as t goes to $+\infty$, we may assume that f is nondecreasing on the whole $[0, +\infty[$ and that $f(t)$ is bigger than any constant (to be decided later) for all t in $[0, +\infty[$, so in particular that f is positive.

Let $\mathcal{T}$ be a rooted tree, with T its set of vertices and x its root. For $n \in \mathbb{N}$, we denote by T_n the set of vertices at distance n from the root. Define the *parent* of $v \in T_{n+1}$ as the unique u in T_n which is joined by an edge to v. We call a *child* of $u \in T_n$ to be any element of the subset $T(u)$ of vertices in T_{n+1} joined by an edge to u.

For every u in X, define $\mathcal{B}_*(u) = \partial X$ if $u = x$ and if $u \neq x$ then let

$$\mathcal{B}_*(u) = \mathcal{B}(u_\infty, e^{-[d(x,u)+f(d(x,u))]}).$$

The following proposition is crucial. It implies the existence of a tree whose ends give rise to a large Cantor type subset of $\mathcal{O}_f$.

Proposition 5.2. *If Γ is cocompact, there exist a rooted tree $\mathcal{T}$ with root x and whose other vertices are in Γy, and a constant $c > 0$ such that*

(1) *if v is a child of u, then $\mathcal{B}_*(v)$ is contained in $\mathcal{B}_*(u)$,*

(2) *if v is a child of u, then $f\big(d(x,u)\big) \leqq d(x,v) - d(x,u) \leqq f\big(d(x,u)\big) + c$,*

(3) *if v, w are children of u, then $2\mathcal{B}_*(v)$ and $2\mathcal{B}_*(w)$ are disjoint,*

(4) *for every vertex u, we have $\sum\limits_{v \in T(u)} e^{-sd(x,v)} \geqq e^{-s[d(x,u)+f(d(x,u))]}$.*

Proof. We first define the constant c, and we will then define T_n by induction on n.

Definition of the constant c. For $k \in \mathbb{N}$, define A_k as

$$A_k = \{u \in X \mid k \leqq d(x,u) < k+1\}.$$

By [Pau], p. 234, there exist $r_0 > 0, c_4 > 0$ and two distinct points a_+, a_- in ∂X such that, with $\mathcal{B}_\pm = \mathcal{B}(a_\pm, r_0)$,

- for every γ in Γ, one of $\gamma\mathcal{C}\mathcal{B}_+$ or $\gamma\mathcal{C}\mathcal{B}_-$ is contained in the shadow cone of $B(\gamma y, c_4)$,

- $\limsup\limits_{n \to +\infty} e^{-ns} \operatorname{Card}\{\gamma \in \Gamma \mid \gamma y \in \mathcal{C}\mathcal{B}_\pm \cap A_n\} = +\infty$.

Since Γ is cocompact, there exists a constant $R > 0$ such that every open ball of radius R contains a point of the orbit Γy. Define $c_5 = \dfrac{1}{a}\log\big(4\sinh\big(a(R + c_4 + 1)\big)\big)$ which is strictly positive. Define $c_6 = e^{s(R+2c_4+c_5+d(x,y))}$.

By discreteness, there exists $r_1 > 0$ such that two balls of radius r_1 centered at two distinct points of the orbit Γx are disjoint. Up to increasing f on a compact subset of its domain of definition, we may assume that $2e^{-f(t)} \leqq r_1$ for all t in $[0, +\infty[$. By the pinched curvature hypothesis, there exists $N \in \mathbb{N}$ such that every ball of radius $2\big(1 + 2\big(2r_1 + c_4 + d(x,y)\big)\big)$ contains at most N pairwise disjoint balls of radius r_1. Hence, for every $n \in \mathbb{N}$, if $V_{\pm,n}$ is a maximal separated subset of $\mathcal{C}\mathcal{B}_\pm \cap A_n$, then $\operatorname{Card} \mathcal{C}\mathcal{B}_\pm \cap A_n \leqq N \operatorname{Card} V_{\pm,n}$.

Let $n_\pm$ be integers such that, with $V_\pm = V_{\pm,n_\pm}$,

(i) $n_\pm \geqq \sup\{2R + 3c_4 + c_5 + d(x,y), f(0)\}$,

(ii) $e^{-s(n_\pm+1)} \operatorname{Card}\{\gamma \in \Gamma \mid \gamma y \in V_\pm\} \geqq c_6$.

Set $c = R + 2c_4 + c_5 + \sup\{n_-, n_+\} + 1 + d(x,y)$.

Construction for $n = 1$. Let $T_0 = \{x\}$ and (for example) $T_1 = \Gamma y \cap V_+$. Let us check that the conditions (1)–(4) of the proposition are satisfied at step $n = 1$ (i.e. for $u = x$).

The assertion (1) is trivially true since $\mathscr{B}_*(x) = \partial X$. By the definition of c and the positivity of f, one has $f(0) + c \geqq c \geqq n_+ + 1$, and $f(0) \leqq n_+$ by (i) so that the assertion (2) holds. Since $2r_v \leqq r_1$ for all v in X, and since $V_{\pm,n}$ is $(1 + 2r_1)$-separated, the shadow cones $\mathscr{C}\mathscr{O}(2B_v)$ and $\mathscr{C}\mathscr{O}(2B_w)$ (with their cone points removed) are disjoint for every distinct v, w in T_1. By Lemma 3.1, this implies that $2\mathscr{B}_*(v)$ and $2\mathscr{B}_*(w)$ are disjoint for every distinct v, w in T_1, so the assertion (3) holds. Since $T(x) = T_1$ and $c_6 \geqq 1 \geqq e^{-sf(0)}$, the assertion (4) is satisfied.

Assume that T_n is constructed with $n \geqq 1$, and let u be in T_n.

Construction of $T(u)$. For every $t \geqq \sup\{c_2, R + c_4 + 1\}$, by Remark 3.2, the shadow $\mathscr{O}\big(B(u_t, R + c_4 + 1)\big)$ is contained in the visual ball

$$\mathscr{B}\big(u_\infty, (4 \sinh a(R + c_4 + 1))^{\frac{1}{a}} e^{-d(x,u_t)}\big).$$

We may assume that $f(0) \geqq \sup\{c_2, R + c_4 + 1\}$, according to the discussion following the statement of Theorem 5.1. Hence the shadow of $B(u_t, R + c_4 + 1)$ is contained in $\mathscr{B}_*(u)$ if and only if, by definition of $\mathscr{B}_*(u)$,

$$\big(4 \sinh a(R + c_4 + 1)\big)^{\frac{1}{a}} e^{-d(x,u_t)} \leqq e^{-[d(x,u) + f(d(x,u))]}$$

that is, if and only if $t \geqq t_0 = d(x,u) + f\big(d(x,u)\big) + c_5$.

Let γy be an orbit point contained in the ball $B(u_{t_0}, R)$. In particular, by the triangle inequality,

$$(*) \qquad |d(x,u) + f\big(d(x,u)\big) + c_5 - d(\gamma y, x)| = |d(u_{t_0}, x) - d(\gamma y, x)| \leqq R.$$

Let $\ell \in \{+, -\}$ be such that γV_ℓ is contained in the shadow cone of $B(\gamma y, c_4)$.

Define $T(u) = \Gamma y \cap \gamma V_\ell$, which is a finite subset of points of the orbit of y under Γ. Let us check the properties (1)–(4) of Proposition 5.2. Let v be in $T(u)$.

Verification of Property (2). Since $T(u)$ is contained in the shadow cone of the ball $B(\gamma x, c_4)$, one has

$$|d(v,x) - d(v,\gamma y) - d(\gamma y, x)| \leqq 2c_4.$$

Since $n_\ell \leqq d(v, \gamma x) \leqq n_\ell + 1$ and by the triangular inequality, one gets

$$n_\ell - d(x,y) \leqq d(v, \gamma y) \leqq n_\ell + 1 + d(x,y).$$

Hence by the equation $(*)$, one has

$$n_\ell - R - 2c_4 - d(x,y) \leqq d(v,x) - d(u,x) - f\big(d(x,u)\big) - c_5 \leqq R + 2c_4 + n_\ell + 1 + d(x,y).$$

Therefore, by the definition of c and the assumption (i) on $n_\pm$, the assertion (2) of Proposition 5.2 holds. More precisely,

$$0 \leqq d(v,x) - d(u,x) - f\big(d(x,u)\big) \leqq c_\ell = R + 2c_4 + c_5 + n_\ell + 1 + d(x,y).$$

Furthermore, by the positivity of f and again the assumption (i) on $n_\pm$, we have

$$d(v,x) - d(u,x) \geqq c_7 = R + c_4 + c_5.$$

Verification of Property (4). By the definition of c_ℓ and c_6, we have

$$e^{-sd(x,v)} \geqq e^{-s(d(x,u)+f(d(x,u))+c_\ell)} = e^{-s[d(x,u)+f(d(x,u))]}\,\frac{e^{-s(n_\ell+1)}}{c_6}.$$

Hence by summing over the v's in $T(u)$, and by using (ii), we obtain

$$\sum_{v\in T(u)} e^{-sd(x,v)} \geqq e^{-s[d(x,u)+f(d(x,u))]}.$$

Therefore, the assertion (4) is satisfied.

Verification of Property (3). Since $2\mathscr{B}_*(v)$ is contained in $\mathscr{O}(2B_v)$ by Lemma 3.1, and since $2r_v = 2e^{-f(d(x,v))} \leqq r_1$, the assertion (3) is satisfied. Indeed, let v, w be children of u. If some geodesic ray α from x meets $B(v,r_1)$ and $B(w,r_1)$, then α lies in $\mathscr{C}\mathscr{O}B(\gamma y, c_4 + r_1)$ by convexity. Hence, with p_v, p_w the orthogonal projections of v, w on α, one has

$$\begin{aligned} d(v,w) &\leqq d(p_v,p_w) + 2r_1 = |d(p_w,x) - d(p_v,x)| + 2r_1 \\ &\leqq |d(p_w,\gamma y) - d(p_v,\gamma y)| + 2(c_4 + r_1) + 2r_1 \\ &\leqq |d(p_w,\gamma x) - d(p_v,\gamma x)| + 2d(x,y) + 2(c_4 + 2r_1) \\ &\leqq 1 + 2\big(c_4 + d(x,y) + 2r_1\big), \end{aligned}$$

which contradicts the fact that $V_\pm$ is $\big(1 + 2\big(c_4 + d(x,y) + 2r_1\big)\big)$-separated.

Verification of Property (1). Since $T(u)$ is contained in the shadow cone of $B(\gamma y, c_4)$, there exists a point z in $B(\gamma y, c_4)$ on the geodesic segment between x and v. Note that

$$\begin{aligned} d(x,z) &\leqq d(x,u_{t_0}) + d(u_{t_0},\gamma y) + d(\gamma y, z) \leqq t_0 + R + c_4 \\ &= d(x,u) + f\big(d(x,u)\big) + c_5 + R + c_4. \end{aligned}$$

Since f is nondecreasing, and since $d(v,x) - d(u,x) \geqq c_7$, the ball

$$\mathscr{B}_*(v) = \mathscr{B}(v_\infty, e^{-(d(x,v)+f(d(x,v)))})$$

is contained in the ball $\mathscr{B}(v_\infty, e^{-(d(x,u)+f(d(x,u))+c_7)})$. By Lemma 3.1, the latter visual ball is contained in the shadow of $B(z,r)$ with

$$r = e^{-(d(x,u)+f(d(x,u))+c_7)}e^{d(x,z)} \leqq e^{R+c_4+c_5-c_7}$$

which is 1 by the definition of c_7. Therefore $\mathscr{B}_*(v)$ is contained in the shadow of $B(z,1)$, which is contained in the shadow of $B(u_{t_0}, R+c_4+1)$ since

$$d(z,u_{t_0}) \leqq d(z,\gamma y) + d(\gamma y, u_{t_0}) \leqq c_4 + R.$$

Since the shadow of $B(u_{t_0}, R+c_4+1)$ is contained in $\mathscr{B}_*(u)$ by definition of t_0, this proves that $\mathscr{B}_*(v)$ is contained in $\mathscr{B}_*(u)$, therefore the assertion (1) holds.

Letting $T_{n+1} = \bigcup_{u\in T_n} T(u)$, the construction at the step $n+1$ is completed. This ends the proof of Proposition 5.2. □

We identify the set ∂T of ends of the tree T constructed in the Proposition 5.2 with the set of sequences $(u_n)_{n\in\mathbb{N}}$ of vertices of T with $u_{n+1} \in T(u_n)$ and $u_0 = x$.

Proposition 5.3. *For every end $(u_n)_{n\in\mathbb{N}}$ of T, there exists ξ in ∂X such that the sequence of points u_n of X converges to ξ. The map $\partial T \to \partial X$ defined by $(u_n)_{n\in\mathbb{N}} \mapsto \xi$ is an homeomorphism onto its image K, which is a Cantor set contained in $\mathcal{O}_f$.*

Proof. Denote by $\bar{E}$ the closure of a subset E of ∂X. By the property (1) in Proposition 5.2, if $(u_n)_{n\in\mathbb{N}}$ is an end of T, then $\left(\overline{\mathscr{B}_*(u_n)}\right)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets whose diameter for the visual distance tends to 0 (since $f(0) > 0$ and $d(x,u_{n+1}) \geqq d(x,u_n) + f(0)$ by the property (2) in Proposition 5.2). Therefore its intersection contains one and only one point, which is the limit of the points u_n by definition of the topology on $X \cup \partial X$.

Note that by the properties (1) and (3) in Proposition 5.2, the visual balls $2\mathscr{B}_*(u)$, $2\mathscr{B}_*(u')$ are disjoint for $u \neq u'$ in T_n, by an easy induction on n. Define K_n as the (finite) union of the closures of the visual balls $\mathscr{B}_*(u)$ for u in T_n. Hence $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of ∂X. Its intersection K is a Cantor set which is the image of the map $\partial T \to \partial X$ defined in the statement of the proposition.

By the definition of $\mathcal{O}_f$, the Cantor set K is contained in $\mathcal{O}_f$. □

Proposition 5.4. *There exists a probability measure μ on the Cantor set K and a constant $C > 0$ such that $\mu\left(\mathscr{B}(\xi,r)\cap K\right) \leqq Cr^{\frac{s}{1+\beta+\varepsilon}}$ for every ξ in K and $r > 0$.*

Proof. Since $K = \bigcap_{n\in\mathbb{N}} \left(\coprod_{u\in T_n} \mathscr{B}_*(u)\right)$, there exists a probability measure μ on K which is defined by $\mu\left(\mathscr{B}_*(x)\right) = 1$ and if v is a child of u, then

$$\mu\left(\mathscr{B}_*(v)\right) = \frac{e^{-s[d(x,v)+f(d(x,v))]}}{\sum_{w\in T(u)} e^{-sd(x,w)}}\,\mu\left(\mathscr{B}_*(u)\right).$$

By the property (4) in Proposition 5.2 and an easy induction argument, we have

$$(**) \qquad \mu\left(\mathscr{B}_*(u)\right) \leqq e^{sf(0)}e^{-s[d(x,u)+f(d(x,u))]}$$

for every vertex u of T.

For every ξ in K and $r > 0$, let $(u_n)_{n \in \mathbb{N}}$ be a ray in T with $\lim_{n \to +\infty} u_n = \xi$. Since $\{2\mathscr{B}_*(u_n) \mid n \in \mathbb{N}\}$ is a neigbourhood basis of ξ, and ∂X has no isolated point, there exists a first n in $\mathbb{N} - \{0\}$ with $\mathscr{B}(\xi, r)$ not contained in $2\mathscr{B}_*(u_n)$. Let z be a point in

$$\mathscr{B}(\xi, r) - 2\mathscr{B}_*(u_n),$$

therefore

$$d_x(z, \xi) \leqq r \quad \text{and} \quad d_x(z, u_\infty) \geqq 2e^{-[d(x,u_n) + f(d(x,u_n))]}.$$

Since ξ belongs to $\mathscr{B}_*(u_n)$, one has

$$d_x(\xi, u_\infty) \leqq e^{-[d(x,u_n) + f(d(x,u_n))]}.$$

Therefore, by the triangular inequality,

$$r \geqq d_x(z, \xi) \geqq d_x(z, u_\infty) - d_x(u_\infty, \xi) \geqq e^{-[d(x,u_n) + f(d(x,u_n))]}.$$

Since $\limsup_{t \to +\infty} \frac{f(t)}{t} = \beta$, there exists a constant $A > 0$ such that $f(t) \leqq (\beta + \varepsilon)t + A$ for every $t \in [0, +\infty[$. Therefore

$$(***) \qquad r \geqq e^{-A} e^{-(1+\beta+\varepsilon)d(x,u_n)}.$$

By the minimality of n, the visual ball $\mathscr{B}(\xi, r)$ is contained in $2\mathscr{B}_*(u_{n-1})$. Therefore $\mathscr{B}(\xi, r) \cap K$ is contained in $2\mathscr{B}_*(u_{n-1}) \cap K = \mathscr{B}_*(u_{n-1}) \cap K$, by the property (3) in Proposition 5.2. Therefore by equation $(**)$,

$$\mu\big(\mathscr{B}(\xi, r) \cap K\big) \leqq \mu\big(\mathscr{B}_*(u_{n-1}) \cap K\big) \leqq e^{sf(0)} e^{-s(d(x,u_{n-1}) + f(d(x,u_{n-1})))}.$$

By the property (2) in Proposition 5.2, one has

$$e^{-sd(x,u_n)} \geqq e^{-s(d(x,u_{n-1}) + f(d(x,u_{n-1})) + c)},$$

so that $\mu\big(\mathscr{B}(\xi, r) \cap K\big) \leqq e^{s(f(0)+c)} e^{-sd(x,u_n)}$. By equation $(***)$, one has

$$\mu\big(\mathscr{B}(\xi, r) \cap K\big) \leqq e^{s(f(0)+c)} (e^A r)^{\frac{s}{1+\beta+\varepsilon}}.$$

This ends the proof of the proposition by taking $C = e^{s(f(0)+c)} e^{\frac{sA}{1+\beta+\varepsilon}}$. □

Proof of Theorem 5.1. By the easy part of Frostman's Lemma (see [Fro]), the Proposition 5.4 implies that the Hausdorff dimension of the Cantor set K (hence the one of $\mathscr{O}_f$ by Proposition 5.3) is at least $\frac{s}{1 + \beta + \varepsilon}$. Since $s = \delta - 2\varepsilon$, the result follows by letting ε tend to 0. □

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