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Groups of automorphisms of trees and their limit sets

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Abstract. Let T be a locally finite simplicial tree and let $\Gamma \subset \mathrm{Aut}(T)$ be a finitely generated discrete subgroup. We obtain an explicit formula for the critical exponent of the Poincaré series associated with Γ, which is also the Hausdorff dimension of the limit set of Γ; this uses a description due to Lubotzky of an appropriate fundamental domain for finite index torsion-free subgroups of Γ. Coornaert, generalizing work of Sullivan, showed that the limit set is of finite positive measure in its dimension; we give a new proof of this result. Finally, we show that the critical exponent is locally constant on the space of deformations of Γ.

1. *Introduction*

Let G be a discrete group of hyperbolic motions. A remarkable family of finite Borel measures supported on the limit set of G, called a *conformal density*, was constructed by S. J. Patterson and D. Sullivan (cf. [**9**] and [**14**]). Coornaert in [**4**] generalized the Patterson–Sullivan construction to a *quasi-conformal density* for subgroups of a hyperbolic group, in the sense of Gromov. In both cases the Poincaré series associated with the group is an essential tool.

The critical exponent δ of the Poincaré series measures the distribution of orbits under the group in the following sense. Let n_k be the number of orbit points of the subgroup in a ball of radius k inside hyperbolic space or a hyperbolic group. Then we have (cf. [**4, 14**])

$$\delta = \limsup_{k\to\infty} \frac{1}{k} \log n_k \tag{1}$$

and for *convex cocompact* groups, δ is the Hausdorff dimension of the limit set of the subgroup.

Throughout this paper, T will denote a locally finite tree (i.e. the degree of every vertex is finite) and Γ a non-elementary finitely generated discrete subgroup of $\mathrm{Aut}(T)$. Such a tree T is Gromov-hyperbolic, so all the above applies in this setting. Moreover, if Γ is torsion free then it is convex cocompact (cf. Proposition 6.2), allowing us to use the stronger results concerning such groups.

The main goal of this paper is to obtain a formula for δ, and give an independent proof that Λ_Γ is of finite positive measure in dimension δ (cf. Theorem 5.3). We will show that the Hausdorff dimension of Λ_Γ is the unique solution of a finite system of equations (cf. Theorem 5.2), and this solution will also allow us to construct the Patterson–Sullivan measure directly.

We will also show that the Hausdorff dimension is locally constant on the space of deformations of Γ (cf. Theorem 7.2) and is therefore continuous on the space of deformations.

The paper is organized as follows. §2 is devoted to a review of trees and their boundaries. In §3 we recall a construction due to Lubotzky [**8**] which enables us to present any discrete finitely generated torsion free subgroup of $\mathrm{Aut}(T)$ as a *Schottky group*. In §4 we recall some known results on *horospherical distance* and *derivatives* for elements in $\mathrm{Aut}(T)$. The main goal of this section is to prove a contracting theorem for the induced action of the generators of Γ in a certain basis (one of its Schottky bases) on the boundary of T.

§5 is devoted to the proof of the first main result (Theorem 5.3). The main tool is the Perron–Frobenius theorem for positive matrices, together with the monotonicity of the leading eigenvalue as a function of the matrix. This monotonicity itself results from the contractions found in §4. In this section we give a direct proof of the formula for the Hausdorff dimension. In §6, we first show how to prove it more easily using the Sullivan–Coornaert machinery. This requires showing that all such groups Γ are convex cocompact, which we also prove in this section. We give another approach using a Markov process on the boundary. This approach was kindly indicated to us by M. Bourdon. §7 studies the deformation space, proving the second main result (Theorem 7.2).

After a first announcement of our results, Gilles Robert has proved that for a discrete finitely generated group Γ of automorphisms of a tree, the Poincaré series is a rational function, and the largest root of the denominator is the critical exponent. He also proved that the Hausdorff dimension of the limit set is locally constant on the deformation space.

2. *Conventions*

We start with a brief review of trees and their boundaries. A tree T is a connected, simply connected graph, with vertices $\mathcal{V}(T)$ and edges $\mathcal{E}(T)$. Throughout this paper, T will be a locally finite tree, i.e. only finitely many edges emanate from each vertex.

A *path* in T is a finite sequence $v_0, \dots, v_n$ such that $v_i, v_{i+1} \in \mathcal{E}(T)$. A *chain* is a path $v_0, \dots, v_n$ such that $v_i \neq v_{i+2}$, for $i = 0, \dots, n-2$.

For any two vertices x, y in $\mathcal{V}(T)$, there exists a unique chain joining x to y. We denote this chain by $[x, y]$ and we call it the *geodesic* joining x to y. The distance $d(x, y)$ between any two distinct vertices x and y is defined as the number of edges in the chain $[x, y]$ joining x to y. An *infinite geodesic* is a sequence of vertices $\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots$ with the property that $v_i \neq v_{i+2}$, and v_i, v_{i+1} are adjacent for all i. Similarly, a half-infinite geodesic $v_0, v_1, v_2, \dots$ is called a *ray*.

The group $G = \mathrm{Aut}(T)$ of the automorphisms of T is equipped with a topology for which the stabilizers of the vertices in finite sets serve as a fundamental system of compact open neighborhoods of the identity. The following is a classification theorem for elements of G (cf. [**5**, p. 7, Theorem 3.2]).

THEOREM 2.1. *Let g be an automorphism of a tree; then one and only one of the following occurs:*
(1) *inversion, i.e. g stabilizes an edge exchanging its vertices;*
(2) *elliptic, i.e. g stabilizes a vertex;*
(3) *hyperbolic, i.e. g stabilizes no vertex.*

Remarks.
(1) An elliptic element g lying in a discrete subgroup must have finite order.
(2) The group $\mathrm{Aut}(T)$ has no analogs of parabolic hyperbolic motions.
(3) If Γ is any subgroup of G which contains inversions, we can eliminate them by passing to the first barycentric subdivision of T.

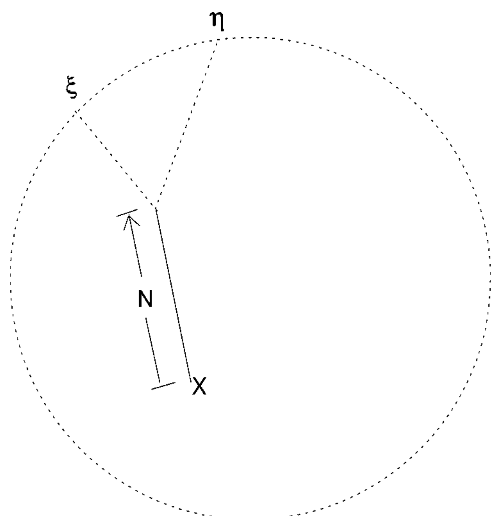

FIGURE 1. Visual distance of points on ∂T.

The *boundary* ∂T is the set of equivalence classes of rays, where two are equivalent if their intersection is infinite; we think of the equivalence class as being a 'point at infinity' at the end of the ray. Given two distinct boundary points $\omega_1, \omega_2 \in \partial T$ there is a unique infinite geodesic connecting them, denoted by (ω_1, ω_2). We also use the notation $[x, \omega)$ for the ray starting at x equivalent to ω, i.e. 'in the direction of ω'. We recall that the space $\bar{T} = T \cup \partial T$ can be given a topology in which it is compact, and T is open and dense in $\bar{T}$. This topology is obtained by defining a basis of neighborhoods for each boundary point. Choose $\omega \in \partial T$, let x be a vertex and let $\tau = [x, \omega)$ be the infinite chain from x to ω. For each $y \in [x, \omega)$ the neighborhood $\mathcal{P}(x, y)$ of ω is defined to consist of all vertices and all end points of the infinite geodesics which include y but no other vertex of $[x, y]$.

Any $\gamma \in \mathrm{Aut}(T)$ extends uniquely to a homeomorphism of $\bar{T}$, and hence acts on ∂T.

In an analogous way to hyperbolic spaces the *visual metric* on the sphere at infinity a family of metrics is defined on ∂T.

Definition 2.2. For each $x \in T$, the *visual metric* on ∂T is defined by the formula

$$|\xi - \eta|_x = e^{-N}, \tag{2}$$

for each $\xi, \eta \in \partial T$, where N is the length of $[x, \xi) \cap [x, \eta)$.

Clearly the visual metric defined above induces on ∂T the topology we described before.

3. *Schottky groups of automorphisms of a tree*

Most of this section is devoted to recalling a construction due to Lubotzky, following work by Gerritzen and van der Put [**6**, Ch. 1]. He showed that finitely generated torsion free subgroups $\Gamma \subset \mathrm{Aut}(T)$ are analogs of classical Kleinian Schottky groups. We also prove Lemma 3.2 which will be used in the proof of Theorem 6.2.

The following lemma, (cf. [**12**, p. 63]) summarizes the properties of hyperbolic elements in G.

LEMMA 3.1. (Tits) *Suppose that* $\gamma \in \mathrm{Aut}(T)$ *is hyperbolic. Set*

$$m = \inf_{x \in \mathcal{V}(T)} d(x, \gamma x) \quad \textit{and} \quad L(\gamma) = \{x \in \mathcal{V}(T) \mid d(x, \gamma x) = m\}. \tag{3}$$

Then we have:

(1) $L(\gamma)$ *is the vertex set of an infinite geodesic in* T*;*
(2) *the action of* g *on* $L(\gamma)$ *induces a translation of magnitude* m*;*
(3) *let* y *be a vertex of distance* l *from* $L(\gamma)$*, then* $d(y, \gamma y) = m + 2l$*.*

The line $L(\gamma)$ is called the *axis* of γ and $m = m(\gamma)$ is called the *translation length* of γ.

LEMMA 3.2. *Let* γ *be a hyperbolic isometry of* $\mathrm{Aut}(T)$*. If* $x \in \mathcal{V}(T)$ *satisfies*

$$[x, \gamma(x)] \cap [\gamma(x), \gamma^2(x)] = \{\gamma(x)\}, \tag{4}$$

then $[x, \gamma(x)] \subset L(\gamma)$*, and the translation length of* γ *equals* $d(x, \gamma(x))$*.*

Proof. If $[x, \gamma(x)]$ is not part of $L(\gamma)$, then $x \notin L(\gamma)$. Let $w \in L(\gamma)$ be the projection of x on $L(\gamma)$; then

$$[\gamma(w), \gamma(x)] \subset [x, \gamma(x)] \cap [\gamma(x), \gamma^2(x)], \tag{5}$$

contradicting the hypothesis. □

Let $\gamma \in \mathrm{Aut}(T)$ be hyperbolic with translation length m and axis L. Following Lubotzky, we choose a labeling of L by symbols x^n, with n an integer, in such a way that $d(x^n, x^{n+1}) = 1$ and $\gamma(x^n) = x^{n+m}$.

Definition 3.3. Set

$$A(\gamma, x^0) = \{x \in \mathcal{V}(T) \mid d(x, x^0) < d(x, x^1)\} \tag{6}$$

and

$$B(\gamma, x^0) = \{x \in \mathcal{V}(T) \mid d(x, x^{m+1}) < d(x, x^m)\}. \tag{7}$$

Definition 3.4. [**8**, Definition 1.4] Let $\gamma_1, \ldots, \gamma_k$ be hyperbolic elements. For $i = 1, \ldots, k$ let $L_i = L(\gamma_i)$ be the axes of the γ_i's. Assume that the L_i's can be labeled in such a way that all the $2k$ subsets $A_i = A(\gamma_i, x_i^0)$ and $B_i = B(\gamma_i, x_i^0)$ are mutually disjoint. The group Γ generated by $\gamma_1, \ldots, \gamma_k$ is called a *Schottky group*.

We will call such a generating set $\gamma_1, \ldots, \gamma_k$ a Schottky basis, and the corresponding A_i, B_i fundamental half-spaces. We will call $x_i = x_i^0$, $y_i = \gamma_i(x_i)$ the *roots* of A_i and B_i respectively.

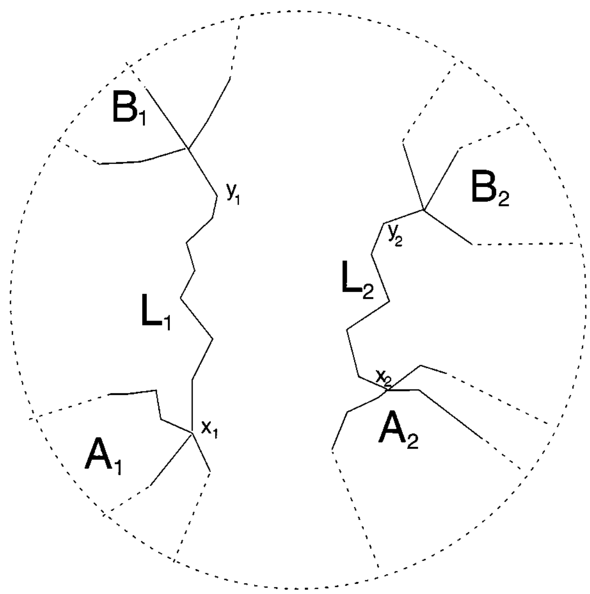

FIGURE 2. Schottky group on 2 generators.

Remark. Figure 2 suggests that the axes L_i are disjoint. This will not usually be the case.

The following proposition presents the basic properties of Schottky groups.

PROPOSITION 3.5. [**8**, Proposition 1.6] *Let Γ be a Schottky group, with Schottky basis $\gamma_1, \ldots, \gamma_k$, and fundamental half-spaces A_i, B_i. Then we have:*

(1) *Γ is a discrete group and every element of Γ is hyperbolic:*

(2) *Γ is a free group with free generators $\gamma_1, \ldots, \gamma_k$:*

(3) *the set $F = \mathcal{V}(T) - \bigcup_{i=1}^{k}(A_i \cup B_i)$ is a fundamental domain for the action of Γ on $\mathcal{V}(T)$.*

The following proposition is essential in all that follows.

PROPOSITION 3.6. [**8**, Proposition 1.7] *Let Γ be a finitely generated torsion free discrete subgroup of* $\mathrm{Aut}(T)$*. Then Γ is a Schottky group.*

We will be interested in general finitely generated discrete subgroups of $\mathrm{Aut}(T)$. The next proposition reveals the importance of Proposition 3.6.

PROPOSITION 3.7. *Let Γ be a finitely generated discrete subgroup of* $\mathrm{Aut}(T)$*. Then Γ has a finite index subgroup Γ' which is torsion free.*

Proof. The proof follows immediately from [**1**, II.8.3] and [**2**, Corollary 2.8 and 4.8]. □

4. *Horospherical distance, derivatives and contracting properties for the induced action on ∂T*

Let Γ be a Schottky group with generators $\gamma_1, \ldots, \gamma_k, k > 1$, and let A_i, B_i be as in Definition 3.4. In this section we prove a contraction property (cf. Theorem 4.7) of the generators in a Schottky group in their induced action on the regions $\partial A_i = \overline{A_i} \cap \partial T, \partial B_i = \overline{B_i} \cap \partial T$. This property plays an essential role in the proof on our first main theorem.

Definition 4.1. The horospherical distance (x, y, ξ) of $x, y \in T$ with respect to $\xi \in \partial T$ is given by

$$(x, y, \xi) = d(x, u) - d(y, u) \tag{8}$$

for any $u \in [x, \xi) \cap [y, \xi)$.

The definition is clearly independent of u.

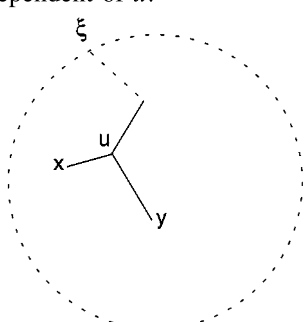

FIGURE 3. Horospherical distance.

If $(x, y, \xi) > 0$, we think of y as being closer to ξ than x. We note that $(x, y, \xi) = 0$ defines an equivalence relation on the vertices of T; the equivalence classes are the *horospheres* based at ξ. The following proposition connects the horospherical distance between vertices in T and the visual metrics on ∂T that they induce (cf. Definition 2.2). We show that all visual metrics are *conformally* the same. For $x_0 \in T, \xi \in \partial T$ and $r > 0$ let $B_{x_0}(\xi, r)$ denote the open ball of radius r around ξ in the visual metric induced by x_0.

PROPOSITION 4.2. *Let $x, y \in T$, and $\xi \in \partial T$. Then there exists $r_0 > 0$ such that for all $0 < r < r_0$ we have*

$$B_x(\xi, r) = B_y(\xi, e^{(x,y,\xi)} r). \tag{9}$$

Proof. Let $[z, \xi) = [x, \xi) \cap [y, \xi)$. It is easy to see that for all $n \geq 0$ we have

$$B_x(\xi, e^{-(d(x,z)+n)}) = B_y(\xi, e^{-(d(y,z)+n)}). \tag{10}$$

This shows that the proposition is true with $r_0 = e^{-(d(x,z)+n)}$. □

A formula for the change of point of observation was obtained in [**4**]:

PROPOSITION 4.3. *Let $x, y \in T$ and $\xi, \eta \in \partial T$. Then*

$$|\xi - \eta|_y^2 = e^{(x,y,\xi)} e^{(x,y,\eta)} |\xi - \eta|_x^2. \tag{11}$$

We next define (cf. [**4**, §3]) the *conformal expansion factor* of $\gamma \in \mathrm{Aut}(T)$, which we will denote as a derivative; it is analogous to the absolute value of the derivative of a hyperbolic isometry, measured in the spherical metric for a chosen base point.

Definition 4.4. Let $\gamma \in \mathrm{Aut}(T)$ be hyperbolic. Let $x \in T$. Then we define the derivative of γ at a boundary point $\xi \in \partial T$ with respect to the induced metric by x on ∂T to be

$$\gamma_x'(\xi) = e^{(x, \gamma^{-1}(x), \xi)}. \tag{12}$$

It follows from [**4**, §3] that a *mean value formula* holds on ∂T.

PROPOSITION 4.5. *Let $\gamma \in \mathrm{Aut}(T)$ be hyperbolic. Let $\xi, \eta \in \partial T$. Then we have*

$$|\gamma(\xi) - \gamma(\eta)|_x^2 = \gamma_x'(\xi)\gamma_x'(\eta)|\xi - \eta|_x^2. \tag{13}$$

Let T be a locally finite tree, and let $\Gamma \subset \mathrm{Aut}(T)$ be a finitely generated discrete torsion free subgroup, which by Proposition 3.6 we know is Schottky for some Schottky basis $\gamma_1, \gamma_2, \ldots, \gamma_k$, with axes L_i, and fundamental half-spaces A_i, B_i, $i = 1, \ldots, k$, with roots x_i, y_i, as in Definition 3.4.

We will work mainly in the set ∂T, and more particularly on the part of ∂T which is not in the closure of the Schottky fundamental domain:

Definition 4.6.

$$C(\Gamma) = \bigcup_{i=1}^{k} (\partial A_i \cup \partial B_i). \tag{14}$$

Give $C(\Gamma)$ the metric $|\xi - \eta|_C$, which on each $\partial A_i, \partial B_i$ is the visual metric coming from its root, and $|\xi - \eta|_C = e$ for ξ, η in distinct fundamental half planes. This metric is conformally equivalent to any visual metric as in Definition 2.2.

The generator γ_i will map each ∂A_j, $j \neq i$ and all ∂B_j into ∂B_i. Denote this restriction by

$$\beta_{j,i} : \partial A_j \to \partial B_i, \ j \neq i, \quad \text{and} \quad \gamma_{j,i} : \partial B_j \to \partial B_i. \tag{15}$$

Similarly, γ_i^{-1} will map each ∂B_j, $j \neq i$ and all ∂A_j into ∂A_i. We denote these maps by

$$\beta_{j,i}' : \partial B_j \to \partial A_i, \ j \neq i, \quad \text{and} \quad \gamma_{j,i}' : \partial A_j \to \partial A_i. \tag{16}$$

In this metric, all the maps above are contracting, with constant contraction factors.

THEOREM 4.7. *For the metric* $|\cdot|_C$, *we have:*
(1) *the mapping* $\beta_{j,i} : \partial A_j \to \partial B_i$ *is contracting with ratio* $b_{j,i} = e^{-d(y_i, \gamma_i(x_j))}$;
(2) *the mapping* $\gamma_{j,i} : \partial B_j \to \partial B_i$ *is contracting with ratio* $c_{j,i} = e^{-d(y_i, \gamma_i(y_j))}$;
(3) *the mapping* $\beta'_{j,i} : \partial B_j \to \partial A_i$ *is contracting with ratio* $b'_{j,i} = e^{-d(x_i, \gamma_i^{-1}(y_j))}$;
(4) *the mapping* $\gamma'_{j,i} : \partial A_j \to \partial A_i$ *is contracting with ratio* $c'_{j,i} = e^{-d(x_i, \gamma_i^{-1}(x_j))}$.

Proof. All these statements are proved in the same way: we show the proof of (3) and leave the rest to the reader. To simplify the notation, set $x = x_i$ and $y = y_j$. Let ξ, η be two points in ∂B_j, $j \neq i$. Suppose first that $L_i \cap L_j = \emptyset$. Let $l_{i,j}$ be the common perpendicular to the axes L_i and L_j. Set $u = L_i \cap l_{i,j}$ and $v = L_j \cap l_{i,j}$. Set ρ to be the length of $l_{i,j}$. By Definition 4.1 we have

$$(x, \gamma_i(x), \xi) = (x, \gamma_i(x), \eta) = d(x, u) - d(\gamma_i(x), u). \tag{17}$$

Using Definition 4.4 we obtain that

$$(\gamma_i^{-1})'_x(\xi) = (\gamma_i^{-1})'_x(\eta) = e^{d(x,u) - d(\gamma_i(x),u)}. \tag{18}$$

We also have that

$$(y, x, \xi) = (y, x, \eta) = -[d(x, u) + \rho + d(y, v)]. \tag{19}$$

By Proposition 4.3 we have

$$|\xi - \eta|_x^2 = e^{(y,x,\xi)} e^{(y,x,\eta)} |\xi - \eta|_y^2. \tag{20}$$

We apply Proposition 4.5 for γ^{-1}, (18), (19) and (20) to obtain

$$|\gamma_i^{-1}(\xi) - \gamma^{-1}(\eta)|_x = e^{-[d(\gamma_i(x),u) + \rho + d(y,v)]} |\xi - \eta|_y. \tag{21}$$

It is easily seen that

$$d(\gamma_i(x), u) + \rho + d(y, v) = d(x, \gamma_i^{-1}(y)). \tag{22}$$

In the case $L_i \cap L_j \neq \emptyset$, we set $u = v = pr\{y\}$ on L_i and continue as above.
This ends the proof. □

5. *The Hausdorff dimension of the limit set*

In this section we prove our first main result.

Definition 5.1. Let $x \in T$, set $\Lambda_\Gamma = \overline{\Gamma \cdot x} \cap \partial T$.

It is easily seen that this definition does not depend on the choice of x. The set Λ_Γ is called the the *limit set* of Γ.

Let $E(\Gamma)$ be the metric space as in Definition 4.6. Let $b_{j,i}, c_{j,i}, b'_{j,i}$ and $c'_{j,i}$ be the constants of contraction defined in Theorem 4.7.

PROPOSITION 5.2. *The system of* $2k+1$ *equations*

$$u_i = \sum_j (c'_{j,i})^s u_i + \sum_{j \neq i} (b'_{j,i})^s v_j \tag{23}$$

$$v_i = \sum_{j \neq i} (b_{j,i})^s u_i + \sum_j (c_{j,i})^s v_j \tag{24}$$

$$1 = \sum_j u_j + \sum_j v_j \tag{25}$$

in the $2k+1$ *variables* $u_i, v_i, i = 1, \ldots, k$ *and* s, *has a unique positive solution.*

Proof. Set

$$P_s = \begin{pmatrix} (c'_{j,i})^s & (b'_{j,i})^s \\ (b_{j,i})^s & (c_{j,i})^s \end{pmatrix}, \quad s > 0. \tag{26}$$

This is a non-negative matrix. A simple calculation shows that P_s^2 has no non-zero entries. By the Perron–Frobenius Theorem (cf. [**13**, Theorem 1.1]), P_s has a unique eigenvector $\mathbf{v}_s$ in the first quadrant, with positive eigenvalue $\lambda(s)$. It is clear that

$$\lim_{s \to \infty} \lambda(s) = 0 \quad \text{and} \quad \lim_{s \to 0} \lambda(s) = 2k - 1. \tag{27}$$

So there exists δ with $\lambda(\delta) = 1$. Moreover, since each of $b_{j,i}, c_{j,i}, b'_{j,i}, c'_{j,i}$ is strictly less than 1, all the entries in P_s are strictly decreasing as functions of s. Therefore, $\lambda(s)$ is strictly decreasing, and δ is unique. The point of introducing the metric $|\cdot|_C$ is to make this true. There is a unique corresponding eigenvector $\mathbf{v}_\delta$ satisfying the last equation, and it has strictly positive entries. □

We will work directly with the definition of Hausdorff δ-measure:

$$\mu_\delta(\Lambda_\Gamma) = \lim_{\epsilon \to 0} \left(\inf \left\{ \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\delta \right\} \right), \tag{28}$$

where the infimum is taken over all covers $\mathcal{U}$ of Λ_Γ by metric balls U with $\operatorname{diam} U < \epsilon$, for the metric $|\cdot|_C$, and δ will be the number found in Proposition 5.2.

THEOREM 5.3. *The limit set* Λ_Γ *is of finite non-zero measure in dimension* δ.

Proof. Consider the sequence of covers $\mathcal{U}_m$ of Λ_Γ, where

$$\mathcal{U}_0 = \{A_1, \ldots, A_k, B_1, \ldots, B_k\} \tag{29}$$

and the set of $\mathcal{U}_{m+1}$ are obtained from those of $\mathcal{U}_m$ by applying the $2k-1$ allowable $\beta_{j,i}, \gamma_{j,i}, \beta'_{j,i}, \gamma'_{j,i}$ to all $U \in \mathcal{U}_m$.

Consider the sequence of vectors $\mathbf{p}_m \in \mathbf{R}^{2k}$, where

$$(\mathbf{p}_m)_i = \sum_{U \in \mathcal{U}_m, U \subset A_i} (\operatorname{diam} U)^\delta, \quad 1 \leq i \leq k, \tag{30}$$

$$(\mathbf{p}_m)_i = \sum_{U \in \mathcal{U}_m, U \subset B_i} (\operatorname{diam} U)^\delta, \quad k+1 \leq i \leq 2k. \tag{31}$$

LEMMA 5.4. *We have* $P_\delta(\mathbf{p}_m) = \mathbf{p}_{m+1}$.

Proof. When $m = 1$, the first line of the matrix identity above says

$$\sum_{U\in\mathcal{U}_1, U\subset A_1} (\operatorname{diam} U)^\delta = \sum_{i=1}^{k} (c'_{i,1})^\delta (\operatorname{diam} A_i)^\delta + \sum_{i=2}^{k} (b'_{i,1})^\delta (\operatorname{diam} B_i)^\delta, \tag{32}$$

which is evidently true since under $\beta'_{i,1}, \gamma'_{i,1}$, the diameters are multiplied by the contraction factors. The proof, in general, is similar. □

LEMMA 5.5. *The sequence* $\mathbf{p}_m = P_\delta^m \mathbf{p}_0$ *converges to a limit* $\mathbf{p}_\infty$ *which is a strictly positive multiple of the eigenvector* v_δ.

Proof. Let E_λ be the generalized eigenspace for the eigenvalue λ, i.e. the subspace corresponding to the Jordan block for eigenvalue λ. The iterates P_δ^m converge to 0 on E_λ when $|\lambda| < 1$. So the linear transformations P_δ^m converge to the projection onto E_1, parallel to the space

$$E = \bigoplus_{|\lambda|<1} E_\lambda. \tag{33}$$

The space E does not intersect the first quadrant, so $P_\delta^m(\mathbf{p}_0)$ converges to a non-zero multiple of $\mathbf{v}_\delta$. □

This proves half of the theorem: the diameters of the elements of the covers $\mathcal{U}_m$ tend to 0 as $m \to \infty$. Since

$$\sum_{U\in\mathcal{U}_m} (\operatorname{diam} U)^\delta = \sum_{i=1}^{2k} (\mathbf{p}_m)_i \to \sum_{i=1}^{2k} (\mathbf{p}_\infty)_i < \infty, \tag{34}$$

we see that the infimum over all covers is certainly finite.

As usual, it is harder to show that the infimum is not zero. Consider first the 'obvious' probability measure μ on Λ_Γ, which assigns to A_i and B_i the ith and $(k+i)$th coordinates of $\mathbf{v}_\delta$, and to a set $U \in \mathcal{U}_{m+1}$ the measure $b_{j,i}^\delta \mu(\beta_{j,i}(U'))$ if $U = \beta_{j,i}(U')$, etc. The construction of such a measure is a standard exercise in measure theory. One way to do this is to consider the algebra $\mathcal{A}$ generated by the characteristic functions χ_U, $U \in \mathcal{U}_n$ for all n. Clearly this algebra is closed under sups and separates points, so it is dense in the algebra of continuous functions $\mathcal{C}(\Lambda_\Gamma)$ by the Stone–Weierstrass theorem. Our construction defines a positive linear functional on $\mathcal{A}$, which extends to $\mathcal{C}(\Lambda_\Gamma)$ by positivity. Now apply the Riesz representation theorem to get the measure μ.

LEMMA 5.6. *Let*

$$K = \inf_{U,V\in\mathcal{U}_1} \inf_{\xi\in U, \eta\in V} |\xi - \eta|_C. \tag{35}$$

Then for any ball $V \subset \Lambda_\Gamma$, *we have*

$$\frac{\mu(V)}{(\operatorname{diam} V)^\delta} \le \frac{1}{K}. \tag{36}$$

Proof. Apply appropriate $\beta_{j,i}$, etc to V until the set V' obtained is no longer contained in a single $U \in \mathcal{U}_1$. The composition leading from V to V' has a certain expansion factor c. Clearly,

$$\frac{c^\delta \mu(V)}{(c \operatorname{diam} V)^\delta} = \frac{\mu(V')}{(\operatorname{diam} V')^\delta} \leq \frac{1}{K}. \tag{37}$$

It follows immediately that for any cover $\mathcal{V}$, we have

$$\sum_{V \in \mathcal{V}} (\operatorname{diam} V)^\delta \geq \frac{1}{K} \sum_{V \in \mathcal{V}} \mu(V) \geq \frac{1}{K}. \tag{38}$$

□

This shows that the infimum in equation (28) is strictly positive and concludes the proof of Theorem 5.3. □

We now consider a general finitely generated subgroup of $\operatorname{Aut}(T)$.

COROLLARY 5.7. *Let Γ be a discrete finitely generated subgroup of* $\operatorname{Aut}(T)$. *Then Λ_Γ is of finite non-zero measure in its dimension.*

Proof. By Proposition 3.7, Γ has a finitely generated torsion free subgroup Γ' of finite index. Clearly, $\Lambda_\Gamma = \Delta_{\Gamma'}$. □

Remark. The Hausdorff dimension of Λ_Γ can be computed as in Theorem 5.3 using Γ'.

6. *Alternative approaches to the Hausdorff dimension*

6.1. *Using the Patterson–Sullivan measures.* There is a shorter path from Proposition 5.2 to Theorem 5.3, using some heavy artillery of Sullivan's, as generalized by Coornaert.

Let E be a closed subset of ∂T. The *convex hull*, $C(E)$, is by definition the smallest convex subset in T whose closure contains E. We note that for a discrete subgroup Γ of $\operatorname{Aut}(T)$, $C(\Lambda_\Gamma)$ is invariant under the action of Γ.

Definition 6.1. Let Γ be a discrete subgroup of $\operatorname{Aut}(T)$. We call Γ *convex cocompact* if $C(\Lambda_\Gamma)/\Gamma$ is compact.

PROPOSITION 6.2. *Let Γ be a finitely generated discrete torsion free subgroup of* $\operatorname{Aut}(T)$. *Then Γ is convex cocompact.*

Proof. By Proposition 3.6, Γ is a Schottky group. We follow the notation of §3. Clearly, $C(\Lambda_\Gamma)/\Gamma$ is a locally finite graph; we will be done if we can show it has finitely many vertices. Since F is a fundamental domain for the action of Γ on $\mathcal{V}(T)$, the canonical projection $F \cap C(\Lambda_\Gamma) \to \mathcal{V}(C(\Lambda_\Gamma)/\Gamma)$ is surjective (in fact, bijective) so it is enough to show that $C(\Lambda_\Gamma) \cap F$ is finite.

Suppose $x \in C(\Lambda_\Gamma) \cap F$. Then $x \in (\xi, \eta)$ for some $\xi, \eta \in \Lambda_\Gamma$; these cannot belong to the same A_i or B_j, since otherwise $(\xi, \eta) \cap F = \emptyset$.

Suppose $\xi \in A_i$ and $\eta \in B_j$ (the other cases are similar). Then $[x_i, y_j] \subset (\xi, \eta)$, and $F \cap (\xi, \eta) = F \cap [x_i, y_j]$. So we see that $C(\Lambda_\Gamma) \cap F$ is contained in the convex hull of $\cup_i \{x_i \cup y_i\}$. The convex hull of finitely many vertices in T is compact. □

We now recall some results on *conformal densities*; the reader is referred to [**4**], [**9**] and [**14**] for the details.

In our setting, a conformal density of dimension δ on ∂T is a rule which associates to every $x \in \mathcal{V}(T)$ a measure μ_x on ∂T, such that $0 < \mu_x(\partial T) < \infty$, and

$$\mu_{\gamma^{-1}x}(E) = \int_E \left(\gamma_x'(\xi)\right)^\delta d\mu_x(\xi), \tag{39}$$

where $\gamma_x'(\xi)$ is as in Definition 4.4.

Patterson and Sullivan have developed the theory of conformal densities for discrete groups of hyperbolic motions, and Coornaert [**4**] generalized this to subgroups of hyperbolic groups in the sense of Gromov. This applies in particular to discrete groups of automorphisms of trees.

Let s be a positive real number, let $x, y \in T$. Set

$$g_s(x, y) = \sum_{\gamma \in \Gamma} e^{-s \cdot d(x, \gamma y)}, \tag{40}$$

$g_s(x, y)$ is called the *Poincaré series* of Γ. The *critical exponent* of Γ is $\delta(\Gamma) = \inf\{s : g_s(x, y) < \infty\}$. It is easily seen that δ is a function of the group only and that $\delta = \limsup(1/k)\log n_k$, where n_k is the number of y-orbit points in a ball of radius k around x.

Let Γ be a finitely generated discrete subgroup of $\mathrm{Aut}(T)$. Theorem 8.3 of [**4**] implies the following.

THEOREM 6.3. *If Γ is convex cocompact, the only Γ-invariant conformal densities on Λ_Γ are the constant multiples of the Hausdorff density of dimension $\delta(\Gamma)$ on the limit set.*

Using this result, we get another proof that the number δ of Proposition 5.2 is the Hausdorff dimension of Λ_Γ.

THEOREM 6.4. *The Hausdorff dimension of Λ_Γ is δ.*

Proof. It is clear that

$$\Lambda_\Gamma \subset E(\Gamma). \tag{41}$$

For each $i = 1, \ldots, k$ the portion of Λ_Γ that is contained in $\partial A_i \cap \Lambda_\Gamma$ is composed of the images of $\partial A_j \cap \Lambda_\Gamma$, $j = 1, \ldots, k$ and $\partial B_j \cap \Lambda_\Gamma$, $j \neq i$ under $\gamma_{i,j}'$ and $\beta_{i,j}'$ respectively. The mappings $\beta_{i,j}'$ and $\gamma_{i,j}'$ have contracting constants $b_{j,i}'$ and $c_{j,i}'$ respectively (see Theorem 4.7).

We know by Theorem 6.3 that if s is the Hausdorff dimension of Λ_Γ, there is a multiple μ_s of Hausdorff s-measure which is a probability measure. Then the numbers

$$\mu_s(\partial A_i \cap \Lambda_\Gamma), \quad \mu_s(\partial B_i \cap \Lambda_\Gamma) \tag{42}$$

satisfy equation (25). Since Hausdorff s-measure transforms under conformal mappings f by multiplication by $|f'|^s$, we have

$$\mu_s(\gamma_{j,i}'(\partial A_j \cap \Lambda_\Gamma)) = (c_{j,i}')^s M_{A_j}, \quad j = 1, \ldots, k \tag{43}$$

$$\mu_s(\beta_{j,i}'(\partial B_j \cap \Lambda_\Gamma)) = (b_{j,i}')^s M_{B_j}, \quad j \neq i. \tag{44}$$

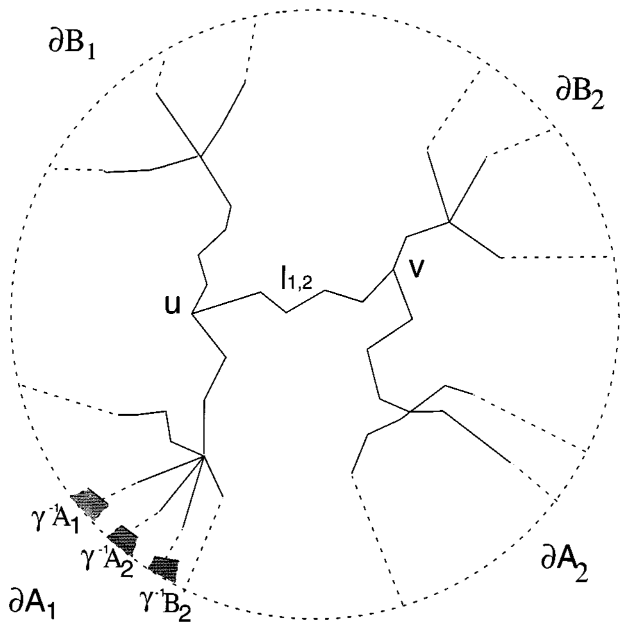

FIGURE 4. The case of two generators.

Since the maps above have disjoint images which cover Λ_Γ, we see that equations (23) and (24) are satisfied as well. So $s = \delta$ by the uniqueness in Proposition 5.2. $\square$

Remarks. We follow the same ideas as in Corollary 5.7 and the Remark following it to get the same conclusion for Γ finitely generated with torsion.

We have not quite constructed a conformal density, because our Hausdorff measure is with respect to the metric $|\cdot|_C$, which does not transform nicely under the group. However, it is easy to recover the conformal density. Choose a point $x \in F$. Then on each A_i and B_i, the visual metric $|\cdot|_x$ is a constant multiple of $|\cdot|_C$, in fact by $e^{-d(x,x_i)}$ and $e^{-d(x,y_i)}$ respectively. So if we set

$$\mu_x = \begin{cases} (e^{-d(x,x_i)})^\delta \mu & \text{on } A_i \\ (e^{-d(x,y_i)})^\delta \mu & \text{on } B_i \end{cases} \tag{45}$$

we get a multiple of Hausdorff measure for the visual metric $|\cdot|_x$, which can be adapted to visual metrics for points $x \notin F$ using equation (39).

6.2. *A Markov approach.* In this subsection we follow Bourdon's ideas to build a Markov process on Λ. This will allow us to use the Perron–Frobenius-Ruelle machinery.

The main ideas are as in the Fuchsian case via Bowen and Series (see [**11**]; basic facts can be found in [**3**], [**7**], [**10**] and [**11**]). In order to keep the same notation as in the above references we have made a small modification in our notation. Call $\gamma_1, \ldots, \gamma_n$ the generators as in Proposition 3.5 and call $\gamma_{-1}, \ldots, \gamma_{-n}$ their inverses. Denote by

$y_{-1}, \dots, y_{-n}$ the roots x_i, by $U_1, \dots, U_n$ the sets $\partial B_i \cap \Lambda$, and by $U_{-1}, \dots, U_{-n}$ the sets $\partial A_i \cap \Lambda$. Now define a Bowen–Series map by

$$f : \Lambda \to \Lambda$$
$$f|_{U_i}(\xi) = \gamma_i^{-1}(\xi).$$

It is well known that f is a Markov map and it is topologically conjugate to a shift on an obvious subshift of finite type. Let x be an origin in T and let φ be the following function on Λ:

$$\begin{aligned} \varphi(\xi) &= \log f_x'(\xi) \\ &= (x, \gamma_i x, \xi), \qquad \text{if } \xi \in U_i. \end{aligned}$$

Then the H-dim of Λ is the unique solution δ of the equation

$$\text{Pressure}\,(-t\varphi) = 0$$

and the H-measure of $(\Lambda, |\cdot|_x)$ is the unique invariant measure of the Perron–Frobenius–Ruelle operator associated to $(\Lambda, f, -\delta\varphi)$.

Theorem 4.7 can be stated as follows. Assume x belongs to the fundamental domain $\mathcal{F}$ as defined in Proposition 3.5. Then if $\xi \in U_i \cap f^{-1}(U_j)$,

$$\varphi(\xi) = d(y_{-i}, y_j) + d(x, y_i) - d(x, y_j). \tag{46}$$

In other words, define the functions u and ψ on Λ by:

$$\begin{aligned} u(\xi) &= d(x, y_i) \qquad \text{if } \xi \in U_i \\ \psi(\xi) &= d(y_{-i}, y_j) \qquad \text{if } \xi \in U_i \cap f^{-1}(U_j), \end{aligned}$$

then (46) implies that

$$\varphi(\xi) = \psi(\xi) + u(\xi) - u(f(\xi)).$$

So φ and ψ are cohomologous, and so the H-dim and the H-measure can be expressed with ψ and u. For example, in the case of the H-dim, we have the following result. Let $A(t)$ the $2n \times 2n$ matrix defined by

$$A(t) = (a_{ij}(t)), \quad i, j \in \{-n, \dots, -1, 1, \dots, n\}$$
$$a_{ij}(t) = \exp(-td(y_{-i}, y_j)) \text{ if } j \neq -i; \quad a_{ij}(t) = 0 \text{ if } j = -i.$$

Then the pressure of $-t\varphi$ is the log of the largest eigenvalue of $A(t)$.

7. *The deformation space*

In this section we prove our second main result.

Definition 7.1. The deformation space of Γ in $\mathrm{Aut}(T)$, is the space of all injective homomorphisms $\rho : \Gamma \to \mathrm{Aut}(T)$ such that $\rho(\Gamma)$ is discrete.

We will now consider the behaviour of the critical exponent of Γ on its space of deformations, when Γ is finitely generated.

Let $\nu_1, \dots, \nu_k$ be generators of Γ. Then the mapping

$$\mathrm{Hom}(\Gamma, \mathrm{Aut}(T)) \to (\mathrm{Aut}(T))^k \tag{47}$$

which sends ρ to $(\rho(\nu_1), \dots, \rho(\nu_k))$ is clearly injective, giving the representation space $\mathrm{Hom}(\Gamma, \mathrm{Aut}(T))$ a topology which does not depend on the choice of generators. Let $\mathcal{R} \subset \mathrm{Hom}(\Gamma, \mathrm{Aut}(T))$ be the subset of injective representations with discrete image.

THEOREM 7.2. *The critical exponent $\delta(\rho(\Gamma))$ is locally constant on $\mathcal{R}$.*

Proof. Let $\Gamma' \subset \Gamma$ be a torsion-free subgroup of finite index, which we can take to be Schottky. Let $\gamma_1, \dots, \gamma_k$ be a Schottky basis of Γ' (using Proposition 3.6 and Proposition 3.7). Let x_i, y_i, $i = 1, \dots, k$ be vertices on L_i as in Definition 3.4. Set ρ_0 to be the injection $\Gamma \to \mathrm{Aut}(T)$, and consider the subset $\mathcal{N} \subset \mathcal{R}$ given by

$$\mathcal{N} = \{\rho \in \mathcal{R} \mid \rho(\gamma_i)(x_i) = y_i, \rho(\gamma_i)(y_i) = \gamma_i(y_i)\}. \tag{48}$$

It is clear that $\rho_0 \in \mathcal{N}$, and that $\mathcal{N}$ is open in $\mathcal{R}$.

We claim that $\rho|_{\Gamma'}$ is Schottky for all $\rho \in \mathcal{N}$, with the same Schottky data as $\rho_0|_{\Gamma'}$. Indeed, by Lemma 3.2 with $x = x_i$, $\gamma = \rho(\gamma_i)$, we deduce that $[x_i, y_i]$ is part of the axis of $\rho(\gamma_i)$ for all $i = 1, \dots, k$, and that is enough to justify the claim.

The computation of the Hausdorff dimension for a Schottky group depends only on the x_i, y_i, and so the Hausdorff dimension of the limit set of $\rho(\Gamma')$ is constant for $\rho \in \mathcal{N}$, and since $\rho(\Gamma')$ is of finite index in $\rho(\Gamma)$, the same holds for the Hausdorff dimension of $\Lambda(\rho(\Gamma))$. □

The reader will observe that we have proved that ρ_0 has a neighborhood in $\mathrm{Hom}(\Gamma, \mathrm{Aut}(T))$ consisting of representations with discrete image.

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